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Sufficiency, ancillarity and completeness from a completely predictive perspective

P. Campagnoli - P.Muliere

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Summary

We discuss the connections between ancillarity, sufficiency and completeness from the notion of conditional independence in a completely predictive framework.

Keywords: Conditional independence, ancillarity, sufficiency, completeness, exchangeable random variables, de Finetti's measure.

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1 Introduction

The concept of conditional independence, well known and very useful in probability theory, becomes more interesting in the theory of statistical inference, where it can be used as a basic tool to express many of the important concepts of statistics, such as sufficiency, ancillarity and completeness (see, for example, Dawid, 1979; Basu and Pereira, 1983; Mouchart e Rolin, 1984; Lloyd, 1988). The purpose of this paper is to investigate in a completely predictive approach the relationship between sufficient, ancillary and complete σ-fields by using the language of conditional independence. The properties of predictive sufficient σ-fields are discussed in Section 3. Under the assumption of exchangeability the equivalence of classical, Bayesian and predictive sufficiency is can be proved (Section 4). The definitions of ancillary σ-field and complete σ-field in a predictive framework, are given in Sections 5 and 6. In particular the Basu's results on the connections of sufficiency, ancillarity and completeness and on the maximal ancillarity are revised in a completely predictive approach (Sections 7 and 8).

2 Notation and preliminaries

Let $(\Omega, \mathcal{F}, P)$ a complete probability space, $\mathcal{F}_0 = \{\emptyset, \Omega\}$ the trivial σ-field, $\bar{\mathcal{F}}_0 = \{A \in \mathcal{F} | P(A) = 0 \text{ o } 1\}$ the completed σ-field of $\mathcal{F}_0$.

We shall denote by $\bar{\mathcal{G}}$ the completed σ-field of $\mathcal{G}$, i.e. the least σ-field containing $\mathcal{G}$ and the P-null sets: $\bar{\mathcal{G}} = \mathcal{G} \vee \bar{\mathcal{F}}_0$.

We define

$$\mathsf{F} = \{\mathcal{G} \text{ sub-}\sigma\text{-fields of } \mathcal{F} | \mathcal{G} = \bar{\mathcal{G}}\}$$

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If $\mathcal{H}, \mathcal{G} \in \mathsf{F}$, then $\mathcal{H} \vee \mathcal{G}$ is the smallest σ-field of F containing $\mathcal{H}$ e $\mathcal{G}$.

For $\mathcal{G} \in \mathsf{F}$

$$L^+(\mathcal{G}) = \{x : \Omega \to \mathbf{R}_+ | x \text{ is } \mathcal{G} - \text{measurable}\}$$

If $x : (\Omega, \mathcal{F}) \to (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))^1$, $\mathcal{F}^x$ denotes the σ-field generated by x.

Recall that $\mathcal{A}_1, \mathcal{A}_2 \in \mathsf{F}$ are independent σ-fields if for any $A_1 \in \mathcal{A}_1$ e $A_2 \in \mathcal{A}_2$, we have:

$$P(A_1 \cap A_2) = P(A_1)P(A_2)$$

or equivalently, if for any $x_1 \in L^+(\mathcal{A}_1)$, $x_2 \in L^+(\mathcal{A}_2)$

$$E[x_1 x_2] = E[x_1]E[x_2]$$

The notation $\mathcal{A}_1 \underline{\|} \mathcal{A}_2$ will be used to indicate that $\mathcal{A}_1, \mathcal{A}_2$ are independent σ-fields. If we want to make explicit the role of the probability P in this concept, we write $\mathcal{A}_1 \underline{\|} \mathcal{A}_2; P$

Now the definition and properties of conditional independence are briefly discussed.

Definition 1 *The conditional independence relation is a relation for a triple of σ-fields $\mathcal{A}_1, \mathcal{A}_2, \mathcal{G} \in \mathsf{F}$ defined by the condition that for all $x_1 \in L^+(\mathcal{A}_1)$, $x_2 \in L^+(\mathcal{A}_2)$:*

$$E[x_1 x_2 | \mathcal{G}] = E[x_1 | \mathcal{G}]E[x_2 | \mathcal{G}] \quad a.s. \tag{1}$$

The notation $\mathcal{A}_1 \underline{\|} \mathcal{A}_2 | \mathcal{G}$ or $\mathcal{A}_1 \underline{\|} \mathcal{A}_2 | \mathcal{G}; P$ will be used

Clearly when $\mathcal{G} = \bar{\mathcal{F}}_0$, Definition 1 corresponds to the usual independence of σ-fields.

In the following some elementary properties of the conditional independence relation are listed:

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[1] $\mathcal{B}(\mathbf{R}^n)$ denotes the Borel σ-field associated to $\mathbf{R}^n$

Proposition 2 (Mouchart and Rolin, 1984; van Putten and van Schuppen, 1985) *Let* $\mathcal{A}_1, \mathcal{A}_2, \mathcal{G} \in \mathsf{F}$. *The following conditions are equivalent:*

(i) $\mathcal{A}_1 \underline{\|} \mathcal{A}_2 | \mathcal{G}$

(ii) $\mathcal{A}_2 \underline{\|} \mathcal{A}_1 | \mathcal{G}$

(iii) for all $x_1 \in L^+(\mathcal{A}_1)$

$$E[x_1 | \mathcal{A}_2 \vee \mathcal{G}] = E[x_1 | \mathcal{G}] \quad a.s.$$

(iv) $\mathcal{A}_1 \vee \mathcal{G} \underline{\|} \mathcal{A}_2 \vee \mathcal{G} | \mathcal{G}$

(v) for all $z \in L^+(\mathcal{A}_1 \vee \mathcal{G})$

$$E\Big[E[z|\mathcal{G}]\Big|\mathcal{A}_2\Big] = E[z|\mathcal{A}_2] \quad a.s.$$

In the following we give two sufficient conditions for a triple of σ-fields to be conditional independent.

Proposition 3 (van Putten and van Schuppen, 1985) *Given* $\mathcal{A}_1, \mathcal{A}_2, \mathcal{G} \in \mathsf{F}$.

a. If $\mathcal{A}_1 \subseteq \mathcal{G}$ *or* $\mathcal{A}_2 \subseteq \mathcal{G}$, *then* $\mathcal{A}_1 \underline{\|} \mathcal{A}_2 | \mathcal{G}$. *In particular* $\mathcal{A}_1 \underline{\|} \mathcal{A}_2 | \mathcal{A}_1$, $\mathcal{A}_1 \underline{\|} \mathcal{A}_2 | \mathcal{A}_2$

b. If $\mathcal{A}_1 \underline{\|} \mathcal{A}_2 \vee \mathcal{G}$, *then* $\mathcal{A}_2 \underline{\|} \mathcal{A}_1 | \mathcal{G}$

The property(iii) of Proposition 2 denotes that, conditionally on $\mathcal{G}$, $\mathcal{A}_2$ does not give useful further information. Therefore the conditional independence is the same as a sufficient property for σ-fields (Skibinski 1967). Also the notion of splitting σ-field introduced by Mckean (1963) is the same as the notion of conditional independence. Therefore will say that a σ-field $\mathcal{G}$ satisfying $\mathcal{A}_1 \underline{\|} \mathcal{A}_2 | \mathcal{G}$ is a splitting σ-field.

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Definition 4 *The notion of minimal conditional independence CI_{min} for a triple of σ-fields $\mathcal{A}_1, \mathcal{A}_2, \mathcal{G} \in \mathsf{F}$ is defined by the conditions*

1. $\mathcal{A}_1 \underline{\|} \mathcal{A}_2 | \mathcal{G}$

2. *if $\mathcal{H} \in \mathsf{F}, \mathcal{H} \subseteq \mathcal{G}$ e $\mathcal{A}_1 \underline{\|} \mathcal{A}_2 | \mathcal{H}$, then $\mathcal{H} = \mathcal{G}$.*

The notation $\mathcal{A}_1 \underline{\|}_m \mathcal{A}_2 | \mathcal{G}$ will be used.

The following definition of projection of a σ-field on another was introduced by Mckean (1963).

Definition 5 *For any two sub-σ-fields $\mathcal{A}_1, \mathcal{A}_2 \in \mathsf{F}$ we define the projection of $\mathcal{A}_1$ on $\mathcal{A}_2$ as follows:*

$$\sigma(\mathcal{A}_1 | \mathcal{A}_2) = (\sigma\{E(x | \mathcal{A}_2) : \forall x \in L^+(\mathcal{A}_1)\} \vee \bar{\mathcal{F}}_0) \in \mathsf{F}$$

We observe that $\sigma(\mathcal{A}_1 | \mathcal{A}_2)$ represents the completed σ-field generated by every version of the conditional expectation of every positive $\mathcal{A}_1$-measurable function.

The motivation to introduce the projection of σ-fields is to construct the smallest sub-σ-field of a σ-field $\mathcal{A}_2$ conditionally on which $\mathcal{A}_2$ becomes independent of another given σ-field $\mathcal{A}_1$. Indeed the following result is known.

Proposition 6 (Mouchart and Rolin, 1984) *For any two sub-σ-fields $\mathcal{A}_1, \mathcal{A}_2 \in \mathsf{F}$:*

i) $\mathcal{A}_1 \underline{\|} \mathcal{A}_2 | \sigma(\mathcal{A}_1 | \mathcal{A}_2)$

ii) *if $\mathcal{H} \in \mathsf{F}, \mathcal{H} \subset \mathcal{A}_2$ e $\mathcal{A}_1 \underline{\|} \mathcal{A}_2 | \mathcal{H} \implies \sigma(\mathcal{A}_1 | \mathcal{A}_2) \subset \mathcal{H}$*

Now we introduce the notions of the weak and strong identification among σ-fields. From a statistical point of view, it may be useful to recall that the concept of identification of a statistical model corresponds to the weak identification of the σ-field generated by the parameter, by the σ-field generated by the observation ; the concept of strong identification corresponds to the concept of bounded completeness in the sens of Lehmann-Scheffé (for more details see Florens, Mouchart and Rolin (1990)).

Definition 7 *Let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{S} \in \mathsf{F}$. We shall say that $\mathcal{A}_2$ is weakly identified by $\mathcal{A}_1$ conditionally on $\mathcal{S}$ if*

$$\sigma(\mathcal{A}_1 \vee \mathcal{S}|\mathcal{A}_2 \vee \mathcal{S}) = \mathcal{A}_2 \vee \mathcal{S} \qquad (2)$$

We write $\mathcal{A}_2 \prec \mathcal{A}_1|\mathcal{S}$

If $\mathcal{S} = \bar{\mathcal{F}}_0$ we say $\mathcal{A}_2$ is weakly identified by $\mathcal{A}_1$ and write $\mathcal{A}_2 \prec \mathcal{A}_1$.

We give in the following elementary properties of weakly identification.

Theorem 8 (Mouchart and Rolin, 1984) *Let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}, \mathcal{B} \in \mathsf{F}$. Then*

i) $\mathcal{A}_2 \prec \mathcal{A}_1$, $\mathcal{A} \subset \mathcal{A}_2 \implies \mathcal{A}_2 \prec \mathcal{A}_1|\mathcal{A}$

ii) $\mathcal{A}_2 \prec \mathcal{A}_1|\mathcal{B}$, $\mathcal{B} \subset \mathcal{A}_1 \implies \mathcal{A}_2 \prec \mathcal{A}_1$

Definition 9 *Let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{S} \in \mathsf{F}$. We shall say that $\mathcal{A}_2$ is strongly identified by $\mathcal{A}_1$ conditionally on $\mathcal{S}$ if*

$$\forall f(X) \in L^+(\mathcal{A}_2 \vee \mathcal{S}) : E[f(X)|\mathcal{A}_1 \vee \mathcal{S}] = 0 \ a.s. \implies f(X) = 0 \ a.s. \qquad (3)$$

We write $\mathcal{A}_2 \prec\prec \mathcal{A}_1|\mathcal{S}$

If $\mathcal{S} = \bar{\mathcal{F}}_0$ we say $\mathcal{A}_2$ is strongly identified by $\mathcal{A}_1$ and write $\mathcal{A}_2 \prec\prec \mathcal{A}_1$.

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The following theorem is useful in order to investigate the connection between strong identification and minimal sufficiency.

Theorem 10 (Mouchart and Rolin, 1984)

Let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{S} \in \mathsf{F}$ and $\mathcal{S} \subset \mathcal{A}_1$.

If $\mathcal{A}_1 \underline{\parallel} \mathcal{A}_2 | \mathcal{S}$ and $\mathcal{S} \prec\prec \mathcal{A}_2$, then $\sigma(\mathcal{A}_2 | \mathcal{A}_1) = \mathcal{S}$

The strong identification implies the weekly identification.

Corollary 11 *If $\mathcal{A}_1 \prec\prec \mathcal{A}_2 | \mathcal{S}$, then $\mathcal{A}_1 \prec \mathcal{A}_2 | \mathcal{S}$*

In order to investigate the relationship between conditional independence, sufficiency and ancillarity, it is important to introduce the concept of measurable separability of σ-fields.

Definition 12 *Let $\mathcal{A}_1, \mathcal{A}_2 \in \mathsf{F}$.*

$\mathcal{A}_1$ and $\mathcal{A}_2$ are measurably separated (and we use the notation $\mathcal{A}_1 \| \mathcal{A}_2$) if the only events in common are trivial, that is if

$$\mathcal{A}_1 \cap \mathcal{A}_2 = \bar{\mathcal{F}}_0$$

or, equivalently, if

$$\forall x \ \mathcal{A}_1\text{-measurable such that } E(x | \mathcal{A}_2) = x \text{ a.s.} \Longrightarrow x = cost \text{ a.s.}$$

3 Predictive sufficient σ-fields

In the completely predictive approach of inference it is possible to recover some fundamental concepts of classical statistical theory, possibly reformulated, as happens for the concepts of sufficient statistics. Predictive sufficiency

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and its properties have been investigated in many papers among which: Cifarelli and Regazzini (1980,1981, 1982); Campanino and Spizzichino (1981); Dawid (1982); Secchi (1987); Muliere and Secchi (1992)). Related notions of sufficiency have been studied by Lauritzen (1984, 1988), Diaconis and Freedman (1984).

Let $(\Omega, \mathcal{F}, P)$ a completed probability space. Let $(y_t)_{t \in T}$ a stochastic process defined on $(\Omega, \mathcal{F}, P)$, with $T \subseteq \mathbf{R}$. For $s \in T$ we define

$\mathcal{B}_s^- = \bigcap_{n \geq 1} \sigma\{y_t, t < s + \frac{1}{n}\}$ the past of the process (y_t)

$\mathcal{B}_s^+ = \sigma\{y_t, t > s\}$ the future of the process (y_t).

We assume that $\mathcal{B}_s^-$, $\mathcal{B}_s^+$ are completed with respect to the P-null sets.

Definition 13 *A sub-σ-field $\mathcal{S}_s$ of $\mathcal{B}_s^-$ is said to be predictive sufficient if $\mathcal{B}_s^-$ and $\mathcal{B}_s^+$ are conditionally independent given $\mathcal{S}_s$, i.e. $\mathcal{B}_s^- \perp\!\!\!\perp \mathcal{B}_s^+ | \mathcal{S}_s$*

From Proposition 2 an equivalent definition is:

$$P(B|\mathcal{B}_s^- \vee \mathcal{S}_s) = P(B|\mathcal{S}_s) \quad a.s. \quad \forall B \in \mathcal{B}_s^+ \tag{4}$$

Using $\mathcal{S}_s \subseteq \mathcal{B}_s^-$, (4) becomes

$$P(B|\mathcal{B}_s^-) = P(B|\mathcal{S}_s) \quad a.s. \quad \forall B \in \mathcal{B}_s^+ \tag{5}$$

i.e. $\mathcal{S}_s$ contains all the relevant information about the past $\mathcal{B}_s^-$ which is needed to determine the probability of future events in $\mathcal{B}_s^+$. We can observe that $\mathcal{B}_s^-$ predictive sufficient σ-field , but it is too large for our purposes. Evidently we are interested to determine predictive sufficient σ-fields $\mathcal{S}_s \subset \mathcal{B}_s^-$ which are small as possible. This approach allows to the notion of minimal predictive sufficient σ-field.

Definition 14 *A sub-σ-field $\mathcal{S}_s$ of $\mathcal{B}_s^-$ is said to be a minimal predictive sufficient σ-field if $\mathcal{B}_s^- \underline{\|}_{min} \mathcal{B}_s^+ | \mathcal{S}_s$*

Proposition 15 *The projection σ-field $\mathcal{P}_s = \sigma(\mathcal{B}_s^+ | \mathcal{B}_s^-)$ is a predictive sufficient σ-field. Moreover $\mathcal{P}_s$ is unique.*

Proof

That $\mathcal{P}_s$ is a minimal predictive sufficient σ-field is proved in Mouchart and Rolin, 1984. (Theorem 4.3).

$\mathcal{P}_s$ is unique. Suppose that exists $\mathcal{A}_s$ minimal predictive sufficient σ-field contained in $\mathcal{B}_s^-$ such that $\mathcal{B}_s^- \| \mathcal{B}_s^+ | \mathcal{A}_s$. Property of minimality of $\mathcal{P}_s$ gives $\mathcal{P}_s \subseteq \mathcal{A}_s$, but $\mathcal{A}_s$ is minimal predictive sufficient, therefore $\mathcal{A}_s \subseteq \mathcal{P}_s$.$\diamond$

The following properties are almost immediate.

Proposition 16 *$\mathcal{P}_s$ has the following properties:*

i) For any sub-σ-field Λ_s of $\mathcal{B}_s^-$ containing $\mathcal{P}_s$:

$$P(B|\Lambda_s) = P(B|\mathcal{P}_s) \quad a.s. \quad \forall B \in \mathcal{B}_s^+$$

ii) All sub-σ-fields of $\mathcal{B}_s^-$ containing $\mathcal{P}_s$ are predictive sufficient, i.e. if Λ_s is a sub-σ-field such that $\mathcal{P}_s \subseteq \Lambda_s \subseteq \mathcal{B}_s^-$:

$$P(B|\Lambda_s) = P(B|\mathcal{B}_s^-) \quad a.s. \quad \forall B \in \mathcal{B}_s^+$$

iii) $\mathcal{P}_s \supseteq (\mathcal{B}_s^+ \cap \mathcal{B}_s^-)$

Proof

(i) $\forall M \in \Lambda_s \ \forall B \in \mathcal{B}_s^+ \ E(I_M I_B) = E[E(I_M I_B | \mathcal{P}_s)]$. Since $\mathcal{P}_s$ is predictive sufficient and $\Lambda_s \subseteq \mathcal{B}_s^- \ E[E(I_M I_B | \mathcal{P}_s)] = E[P(M|\mathcal{P}_s)P(B|\mathcal{P}_s)]$. From the basic

properties of the conditional expectation, being $P(B|\mathcal{P}_s)$ a $\mathcal{P}_s$-measurable random variable, we get $E[P(M|\mathcal{P}_s)P(B|\mathcal{P}_s)] = E[I_M P(B|\mathcal{P}_s)]$. Therefore $E(I_M I_B) = E[I_M P(B|\mathcal{P}_s)]$ $\forall M \in \Lambda_s$ $\forall B \in \mathcal{B}_s^+$.

(ii) and (iii) are proved in Mouchart e Rolin (1984).$\Diamond$

The problem of characterization of $\mathcal{P}_s$ may be handled in the form of necessary and sufficient conditions for a predictive sufficient σ-field to be minimal.

Theorem 17 *A predictive sufficient σ-field $\mathcal{S}_s$ is minimal (i.e $\mathcal{S}_s = \mathcal{P}_s$) if and only if $\mathcal{S}_s \prec \mathcal{B}_s^+$*

Proof

(necessity) If $\mathcal{S}_s$ is a minimal predictive sufficient σ-field, then, from the uniqueness of $\mathcal{P}_s$, we get $\mathcal{S}_s = \mathcal{P}_s$. Furthermore $\mathcal{P}_s$ is weakly identified by $\mathcal{B}_s^+$. Indeed $\mathcal{P}_s = \sigma(\mathcal{B}_s^+|\mathcal{B}_s^-) = \sigma(\mathcal{B}_s^+|\mathcal{P}_s)$ (see Mouchart and Rolin, 1984; Corollary 4.9).

(sufficiency) Suppose that $\mathcal{S}_s$ is predictive sufficient and $\mathcal{S}_s \prec \mathcal{B}_s^+$. We want to show that $\mathcal{S}_s$ is minimal predictive sufficient, i.e. $\mathcal{S}_s = \mathcal{P}_s$. The conditions (i) $\mathcal{S}_s$ predictive sufficient and (ii) $\mathcal{S}_s \prec \mathcal{B}_s^+$ imply that (iii) $\mathcal{S}_s \prec\prec \mathcal{B}_s^-$. Let $\mathcal{M}_s \subseteq \mathcal{S}_s$ and satisfies $\mathcal{B}_s^+ \| \mathcal{B}_s^- | \mathcal{M}_s$. The condition (iii), being $\mathcal{M}_s \subseteq \mathcal{S}_s$, implies (see Mouchart and Rolin, 1986; A.11(iii)(b)) that (iv) $\mathcal{S}_s \prec\prec \mathcal{B}_s^- | \mathcal{M}_s$. Because $\mathcal{M}_s \subseteq \mathcal{S}_s$, $\mathcal{M}_s$ and $\mathcal{S}_s$ are predictive sufficient σ-fields and the condition (iv) is satisfied, we have (see Mouchart e Rolin, 1986; A.12) $\mathcal{B}_s^+ \| (\mathcal{B}_s^- \vee \mathcal{S}_s) | \mathcal{M}_s$. Since $\mathcal{S}_s \subseteq (\mathcal{B}_s^- \vee \mathcal{S}_s)$, we have $\mathcal{B}_s^+ \| \mathcal{S}_s | \mathcal{M}_s$ and, being $\mathcal{M}_s \subseteq \mathcal{S}_s$, we get (see Mouchart and Rolin, 1986; A.6(ii)) $\sigma(\mathcal{B}_s^+|\mathcal{S}_s) \subseteq \mathcal{M}_s$. But being by supposition $\sigma(\mathcal{B}_s^+|\mathcal{S}_s) = \mathcal{S}_s$, we have $\mathcal{S}_s \subseteq \mathcal{M}_s$.$\Diamond$

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4 Relations between classical sufficiency, Bayesian sufficiency and predictive sufficiency

In this section we formulate the definition of predictive sufficient σ-field, when Ω is a Polish space. We prove an asymptotic result for a sequence of predictive sufficient σ-fields and we investigate the connections between classical sufficiency, Bayesian sufficiency and predictive sufficient σ-fields when the process $y = (y_n)$ is an exchangeable sequence. Letta (1981) proves that a classical sufficient statistic is always a Bayesian sufficient statistic, but the converse is true only if the σ-fields are separable. Fortini, Ladelli, Regazzini (2000)) prove the equivalence between classical , Bayesian and predictive sufficient statistics under the hypotheses of exchangeability.

Let (X, χ) be a measurable space. Write X^n for the n-fold Cartesian product and χ^n for the usual product σ-field $(n = 1, 2, \ldots, \infty)$. Define $y_1, y_2, \ldots$ to be the coordinate random variables (r.v.'s) of X^∞, i.e. $y_i(x) = x_i$ for every $x = (x_1, x_2, \ldots)$ in X^∞. Let M the set of probability measures on (X, χ). M is made into a measurable space by the σ-field $\mathcal{M}$ generated by all sets $\{p \in \mathcal{M} : p(A) \in B\}$ with A in χ and B in $\mathcal{B}([0,1])$. If we assume that X is a Polish space [separable, complete, metric space], then the same holds for M and we can take $\chi = \mathcal{B}(X)$ and $\mathcal{M} = \mathcal{B}(M)$. Let P a exchangeable probability measure on (X, χ). Recall that, under suitable conditions about (X, χ), P admits the *de Finetti representation*.

If X is a Polish space, then the following statements are equivalent

(S1) *$y_1, y_2, \ldots$ are exchangeable whit respect to P*

(S2) *There is a random probability measure $\tilde{p}$ [$= r.v$ on $(X^\infty, \mathcal{B}(X^\infty))$, tak-*

ing values in $(M, \mathcal{B}(M))]$ such that $\tilde{p}^\infty(A)$ is a version of the conditional probability $P\{y \in B | \tilde{p}\}$ for every A in $\mathcal{B}(X^\infty)$, where $\tilde{p}^\infty$ is the power probability measure which makes the coordinates i.i.d. [independent and identically distributed] with probability distribution $\tilde{p}$.

(S3) There exists a unique probability measure ν on $(M, \mathcal{B}(M))$ such that

$$P\{y \in A\} = \int_M p^\infty(A)\nu(dp) \qquad (A \in \mathcal{B}(X^\infty))$$

This measure is said to be the de Finetti measure and coincides with the probability distribution of $\tilde{p}$.

In particular the equivalence of $(S1)$ and $(S2)$ implies that y_n are exchangeable if and only if they are conditionally i.i.d. given the σfield $\sigma(\tilde{p})$.

In the same setting as in Section 3, with $(\Omega, \mathcal{F}, P) \equiv (X^\infty, \mathcal{B}(X^\infty), P)$ let $\mathcal{B}_n^- = \sigma\{y_{(n)}\}$ [where $y_{(n)} := (y_1, \ldots y_n)$] the past of the sequence $y = (y_n)_{n \geq 1}$ and $\mathcal{B}_n^+ = \sigma\{y_i, i \geq n+1\}$ the future of the process $y = (y_n)_{n \geq 1}$.

If we assume that X is a Polish space the definition of predictive sufficient σ-field can be formulated as the following.

Definition 18 *A sub-σ-field $\mathcal{S}_n$ of $\mathcal{B}_n^-$ is said to be predictive sufficient w.r.t P if there is a regular conditional probability distribution (r.c.p.d.) of $(y_i)_{i \geq n+1}$ given $\mathcal{S}_n$ w.r.t. P, say μ_n*[2] *such that, such that $\omega \to \mu_n(\omega, A)$ is a version of $P\{(y_i)_{i \geq n+1} \in A | \mathcal{B}_n^-\}$ for every $A \in \mathcal{B}(X^\infty)$*

We now recall some facts on tail σ-fields.

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[2]By definition of r.c.p.d. $\mu_n : X^\infty \times \mathcal{B}(X^\infty) \to [0,1]$ is such that

(i) $\forall A \in \mathcal{B}(X^\infty)$ $\quad \omega \to \mu_n(\omega, A)$ is a version of $P\{(y_i)_{i \geq n+1} \in A | \mathcal{S}_n\}$

(ii) $\forall \omega \in X^\infty$ $\quad A \to \mu_n(\omega, A)$ is a probability measure on $(X^\infty, \mathcal{B}(X^\infty))$

Definition 19 *Let (Λ_n) be a sequence of sub-σ-fields of $\mathcal{B}_n^-$. The tail σ-field Λ_T is defined as*

$$\Lambda_T = \bigcap_{n \geq 0} \bigvee_{m \geq n} \Lambda_n$$

When Λ_n is non-increasing (i.e., $\Lambda_{n+1} \subset \Lambda_n \ \forall n \in \mathbf{N}$ or $\Lambda_n \downarrow$) $\Lambda_T = \bigcap_{n \geq 0} \Lambda_n$, whereas when Λ_n is non-decreasing-or equivalently a filtration-(i.e., $\Lambda_n \subset \Lambda_{n+1} \ \forall n \in \mathbf{N}$ or $\Lambda_n \uparrow$), $\Lambda_T = \Lambda_\infty = \bigvee_{n \geq 0} \Lambda_n$.

Theorem 20 *Let $y = (y_n)_{n \geq 1}$ be the sequence of coordinate r.v.'s of X^∞. Let $(\mathcal{S}_n)$ a sequence of sub-σ-fields of $\mathcal{B}_n^-$ and let $\mathcal{S}_n$ be a predictive sufficient σ-field for every $n \in \mathbf{N}$. Then*

$$\lim_{n \to \infty} P\{B|\mathcal{S}_n\} = P\{B|S_T\} \quad P q.c. \ \forall B \in \mathcal{B}_n^+$$

where S_T is the tail σ-field correspondent to $(\mathcal{S}_n)$

Proof

By definition of predictive sufficiency $\forall B \in \mathcal{B}_n^+$

$$P(B|\mathcal{B}_n^-) = P(B|\mathcal{S}_n) \quad P \ q.c. \tag{6}$$

Let $f_n := P(B|\mathcal{S}_n)$. f_n is a $\mathcal{B}_n^-$-martingale. By a limit theorem for martingales (see Dellacherie-Meyer, 1980) we have

$$P(B|\mathcal{B}_n^-) \to P(B|\mathcal{B}_\infty^-) \quad P \ q.c.$$

i.e.

$$P(B|\mathcal{B}_n^-)(\omega) \to P(B|\mathcal{B}_\infty^-)(\omega)$$

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for every $\omega \in G(B)^c$ where $G(B) \in \mathcal{B}(X^\infty)$ and $P(G(B)) = 0$. By separability of X it can be proved the existence of a set $G \in \mathcal{B}(X^\infty)$ with $P(G) = 0$, such that

$$P(B|\mathcal{B}_n^-)(\omega) \to P(B|\mathcal{B}_\infty^-)(\omega) \tag{7}$$

for every $\omega \in G^c$ and for every $B \in \mathcal{B}_n^+$.

Taking the limsup on both sides of (6) we have, from (7)

$$P(B|\mathcal{B}_\infty^-) = \text{limsup}_n P(B|\mathcal{S}_n) \quad \text{q.c.} \quad \forall B \in \mathcal{B}_n^+$$

Therefore $P(B|\mathcal{B}_\infty^-)$ is $\mathcal{S}_T$-measurable and $P(B|\mathcal{B}_\infty^-) = P(B|\mathcal{S}_T)$. $\Diamond$

We now investigate the connections between predictive, Bayesian and Fisher's classical sufficiency in terms of σ-fields. It is simple to verify in terms of σ-fields, the results proved for statistics by Fortini, Ladelli and Regazzini (1998, 2000). So we omit the proofs of the following theorems which can be easily obtained by the results of Fortini, Ladelli and Regazzini.

Definition 21 *A sub-σ-field $\mathcal{S}_n$ of $\mathcal{B}_n^-$ is said to be a Bayesian sufficient σ-field w.r.t. P if there is a r.c.p.d. of $\tilde{p}$ given $\mathcal{S}_n$ w.r.t. P, say Q_n' [3], such that $Q_n'(\cdot, C)$ is a version of $P\{\tilde{p} \in C|\mathcal{B}_n^-\}$ for every $C \in \mathcal{B}(M)$.*

Definition 22 *Let $(\Omega, \mathcal{F})$ be a measurable space. A sub-σ-field $\mathcal{S} \subset \mathcal{F}$ is said to be a classical sufficient σ-field for an arbitrary class $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{F})$ if, for every $A \in \mathcal{F}$, there exists a common version of*

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[3] By definition $Q_n' : X^\infty \times \mathcal{B}(M)) \to [0,1]$ is such that

(i) $\forall \omega \in \mathcal{B}(X^\infty)$ $\quad Q_n'(\omega, \cdot)$ is a probability measure on $(M, \mathcal{B}(M))$

(ii) $\forall C \in \mathcal{B}(M)$ $\quad Q_n'(\cdot, C)$ is a version of $P\{\tilde{p} \in C|\mathcal{S}_n\}$

the conditional probabilities $P(A|\mathcal{S})$ $\forall P \in \mathcal{P}$.

If Ω is a Polish space, $\mathcal{S}$ is said to be a classical sufficient σ-field if all $P \in \mathcal{P}$ have a common conditional distribution relative to $\mathcal{S}$; in other words, if for each $\omega \in \Omega$, there exists a probability measure $Q(\omega, \cdot)$ on $\mathcal{F}$ such that, for each A, $Q(\cdot, A)$ is $\mathcal{F}$-measurable and

$$P(A|\mathcal{S}) = Q(\cdot, A) \quad a.s. \ P \quad for\ all\ P \in \mathcal{P}$$

The following theorem states that predictive sufficiency is the same of Bayesian sufficiency.

Theorem 23 *Let the sequence* $y = (y_n)_{n \geq 1}$ *of coordinate r.v.'s of* X^∞ *be exchangeable w.r.t.* P. *Then, w.r.t.* P, $\mathcal{S}_n$ *is a Bayesian sufficient* σ-*field if, and only if, it is a predictive sufficient* σ-*field.*

We obtain, when P is exchangeable, the equivalence between classical sufficiency and predictive sufficiency.

Theorem 24 *Let* P *be an exchangeable probability measure on* $(X^\infty, \mathcal{B}(X^\infty))$ *and let* $\mathcal{S}_n$ *be a predictive sufficient* σ-*field w.r.t.* P. *Then there is a set* N *in* $\mathcal{B}(M)$ *with* $\nu(N) = 0$ *[ν is the de Finetti measure] such that* $\mathcal{S}_n$ *is a classical sufficient* σ-*field for* $\mathcal{P} = \{p^n; p \in N^c\}$; *in other words* $y_{(n)}$ *have a common r.c.p.d. relatively to* $\mathcal{S}_n$, *w.r.t.* p^∞, *for every* $p \in N^c$

Theorem 25 *Let the sequence* y *of coordinate r.v.'s of* X^∞ *be exchangeable w.r.t.* P. *Moreover, let* S_n *be a classical sufficient* σ-*field for* $\mathcal{P} = \{p^n; p \in N^c\}$, *where* $N \in \mathcal{B}(M)$ *with* $\nu(N) = 0$. *Then* $\mathcal{S}_n$ *is a predictive sufficient* σ-*field w.r.t.* P.

5   Predictive ancillarity

The concepts of sufficiency and ancillarity are complementary. If the rest of data is discarded, a predictive sufficient σ-field retains all the information about the past which is needed to determine the probability of future events in $\mathcal{B}_s^+$. A predictive ancillary σ-field contains no information about future events.

Definition 26 *A sub-σ-field Σ_s of $\mathcal{B}_s^-$ is said to be predictive ancillary if $\mathcal{B}_s^+ \| \Sigma_s$, i.e. $P(B|\Sigma_s) = P(B) \ \forall B \in \mathcal{B}_s^+$*

It easy to verify the following properties:

- All sub-σ-fields of $\mathcal{B}_s^-$ which are smaller than Σ_s are predictive ancillary.

- If Σ_s is predictive ancillary and $\Sigma_s \subseteq \mathcal{A}$, then $\mathcal{A}$ is predictive ancillary. (see Mouchart and Rolin, 1984; corollary 2.6)

Definition 27 *A sub-σ-field Σ_s of $\mathcal{B}_s^-$ is said to be maximal predictive ancillary if*

i) Σ_s is predictive ancillary

ii) if $\mathcal{H}$ predictive ancillary and $\Sigma_s \subseteq \mathcal{H}$ then $\Sigma_s = \mathcal{H}$

6   Completeness in a predictive approach

In a predictive approach, the analogous of the concept of completeness is the concept of strong identifiability.

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Definition 28 *A predictive sufficient σ-field $\mathcal{S}_s$ is said to be complete if $\mathcal{S}_s$ is strongly identified by $\mathcal{B}_s^+$, i.e. if*

$$\forall f(X) \in L^+(\mathcal{S}_s) : E[f(X)|\mathcal{B}_s^+] = 0 \ \ a.s. \Longrightarrow f(X) = 0 \ \ a.s. \tag{8}$$

$\mathcal{S}_s$ is said to be boundedly complete if (8) holds for all bounded f.

Lehmann and Scheffé (1950) proved that if a sufficient statistic is boundedly complete, then it is a minimal sufficient statistic. Basu and Pereira (1983) propose a Bayesian version of this result. The proposition below is a predictive version of this result.

Theorem 29 *Suppose $\mathcal{S}_s$ is a complete predictive sufficient σ-field. Then $\mathcal{S}_s$ is a minimal predictive sufficient σ-field, i.e. $\mathcal{S}_s = \mathcal{P}_s$.*

Proof

By assumption $\mathcal{S}_s$ is strongly identified by $\mathcal{B}_s^+$. Then $\mathcal{S}_s$ is weakly identified by $\mathcal{B}_s^+$ (Corollary 11). Then from Theorem 17 we have $\mathcal{S}_s = \mathcal{P}_s$.

7 Basu's theorems in a completely predictive approach

Consider the following three propositions:

(a) $\mathcal{S}_s$ is predictive sufficient σ-field.

(b) Σ_s is a predictive ancillary σ-field.

(c) Σ_s and $\mathcal{S}_s$ are conditional independent given $\mathcal{B}_s^+$

In the classical sampling approach (Basu, 1955, 1958) speculates under what conditions two of three relations (a), (b), (c) imply the third. This problem is studied in the traditional Bayesian approach by Basu and Pereira (1983) and by Florens, Mouchart e Rolin (1990). In this section we study Basu's theorems under the completely predictive approach.

Proposition 30 (Theorem one of Basu (1955))
Suppose $\mathcal{S}_s$ is a complete predictive sufficient σ-field and Σ_s a predictive ancillary σ-field. Then $\Sigma_s \underline{\|} \mathcal{S}_s | \mathcal{B}_s^+$.

Proof
Since by assumption $\mathcal{B}_s^+ \underline{\|} \mathcal{B}_s^- | \mathcal{S}_s$ and $\Sigma_s \subseteq \mathcal{B}_s^-$, we get $\mathcal{B}_s^+ \underline{\|} \Sigma_s | \mathcal{S}_s$, i.e. $\forall x$ Σ_s-measurable

$$E(x | \mathcal{B}_s^+ \vee \mathcal{S}_s) = E(x | \mathcal{S}_s) \quad \text{a.s.} \tag{9}$$

Furthermore $\mathcal{B}_s^+ \underline{\|} \Sigma_s \implies E(x | \mathcal{B}_s^+) = E(x)$ a.s., so, from the properties of the conditional expectation, $E[E(x | \mathcal{S}_s) - E(x) | \mathcal{B}_s^+] = 0$ a.s. Since $\mathcal{S}_s \prec\prec \mathcal{B}_s^+$, we get

$$E(x | \mathcal{S}_s) = E(x) \quad \text{a.s.} \quad \forall x \ \Sigma_s\text{-measurable} \tag{10}$$

(9) and (10)$\implies E(x | \mathcal{B}_s^+ \vee \mathcal{S}_s) = E(x)$ a.s. $\forall x$ Σ_s-measurable, i.e. $\mathcal{S}_s \underline{\|} \Sigma_s | \mathcal{B}_s^+$.$\diamond$

Basu (1955) stated that any statistic independent of a sufficient statistic is ancillary. Later on Basu (1958) presented a counter example and recognized the necessity of an additional condition on the parametric family of probability measures on the sampling space. Koehn and Thomas (1975) strengthened this result by introducing a necessary and sufficient condition on the family. A Bayesian version of the Koehn and Thomas's theorem is contained in Basu and Pereira (1983) and in Mouchart and Rolin (1984).

The following theorem is a completely predictive version of the result of Koehn and Thomas.

Theorem 31 (Theorem two of Basu, 1955)

Let $\mathcal{S}_s$ a predictive sufficient σ-field. Then $\mathcal{B}_s^+$ and $\mathcal{S}_s$ are measurably separated (i.e. $\mathcal{B}_s^+\|\mathcal{S}_s$) if and only if for any sub-σ-field Σ_s of $\mathcal{B}_s^-$ such that $\mathcal{S}_s\underline{\|}\Sigma_s|\mathcal{B}_s^+$, we have that Σ_s is ancillary (i.e. $\Sigma_s\underline{\|}\mathcal{B}_s^+$).

Proof

(Sufficiency) By hypothesis $\mathcal{S}_s$ predictive sufficient σ-field. For any $\Sigma_s \subseteq \mathcal{B}_s^-$, then we get

$$\Sigma_s\underline{\|}\mathcal{B}_s^+|\mathcal{S}_s \tag{11}$$

Furthermore by hypothesis

$$\mathcal{B}_s^+\|\mathcal{S}_s \tag{12}$$

The conditions (11) and (12) imply $\Sigma_s\underline{\|}(\mathcal{B}_s^+\vee\mathcal{S}_s)|(\mathcal{B}_s^+\cap\mathcal{S}_s)$ (see Dawid, 1979). From the definition of separable measurability $\mathcal{B}_s^+\cap\mathcal{S}_s = \bar{\mathcal{F}}_0$, so $\Sigma_s\underline{\|}(\mathcal{B}_s^+\vee\mathcal{S}_s)$. Finally $\mathcal{B}_s^+ \subseteq (\mathcal{B}_s^+\vee\mathcal{S}_s) \implies \Sigma_s\underline{\|}\mathcal{B}_s^+$.

(Necessity) Let $\Sigma_s = \mathcal{B}_s^+\cap\mathcal{S}_s$. Obviously $\Sigma_s \subseteq \mathcal{S}_s \subseteq \mathcal{B}_s^-$. Furthermore $\Sigma_s\underline{\|}\mathcal{S}_s|\mathcal{B}_s^+$. Indeed $\forall A \in \Sigma_s \; \forall B \in \mathcal{S}_s$ we have $P(A\cap B|\mathcal{B}_s^+) = E(I_A I_B|\mathcal{B}_s^+) = I_B E(I_A|\mathcal{B}_s^+) = P(A|\mathcal{B}_s^+)P(B|\mathcal{B}_s^+)$. Then by hypothesis we have $\Sigma_s\underline{\|}\mathcal{B}_s^+$. But $\Sigma_s \subseteq \mathcal{B}_s^+$ by construction. Therefore $\Sigma_s\underline{\|}\Sigma_s$ and this is the same as to state $\Sigma_s = \bar{\mathcal{F}}_0.\Diamond$

8 Basu's results on maximal ancillarity

In this section we extend Basu's results (1955, 1959) which identify maximal ancillarity as complementary to complete sufficiency. All the definitions of

previous sections may be easily generalized by introducing a conditioning σ-field to each relation stated.

Let $\mathcal{M}$ sub-σ-field of $\mathcal{F}$.

Definition 32 *A sub-σ-field $\mathcal{S}_s$ of $\mathcal{B}_s^-$ is said to be predictive sufficient conditionally on $\mathcal{M}$ if*

$$\mathcal{B}_s^- \underline{\parallel} \mathcal{B}_s^+ | (\mathcal{S}_s \vee \mathcal{M})$$

Definition 33 *A sub-σ-field Σ_s of $\mathcal{B}_s^-$ is said to be predictive ancillary conditionally on $\mathcal{M}$ if*

$$\mathcal{B}_s^+ \underline{\parallel} \Sigma_s | \mathcal{M}$$

Definition 34 *A predictive sufficient σ-field $\mathcal{S}_s$ is said to be complete conditionally on $\mathcal{M}$ if*

$$\mathcal{S}_s \prec\prec \mathcal{B}_s^+ | \mathcal{M}$$

The theorems in the following extend the Basu's results (1955;1959) in a predictive approach. For similar results in the traditional Bayesian approach c.f. Lloyd (1988), Mouchart and Rolin(1989), Florens, Mouchart and Rolin (1990).

Theorem 35 *Let $\mathcal{S}_s$ complete predictive sufficient conditionally on $\mathcal{M}$. Let Σ_s predictive ancillary conditionally on $\mathcal{M}$. Then $\mathcal{S}_s \underline{\parallel} \Sigma_s | \mathcal{M}$*

Proof

Since $\Sigma_s \subset \mathcal{B}_s^-$ and $\mathcal{S}_s$ is predictive sufficient conditionally on $\mathcal{M}$, we have

$$E(W | \mathcal{M} \vee \mathcal{B}_s^+ \vee \mathcal{S}_s) = E(W | \mathcal{M} \vee \mathcal{S}_s) \quad \text{a.s.} \quad \forall W \ \Sigma_s - \text{measurable} \quad (13)$$

Since Σ_s is predictive ancillary conditionally on $\mathcal{M}$, we have

$$E(W | \mathcal{M} \vee \mathcal{B}_s^+) = E(W | \mathcal{M}) \quad \text{a.s.} \quad \forall W \ \Sigma_s - \text{measurable} \quad (14)$$

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(13) and (14), together with $\mathcal{B}_s^+ \subseteq \mathcal{M} \vee \mathcal{B}_s^+ \vee \mathcal{S}_s$, imply

$$E\Big[E(W|\mathcal{M} \vee \mathcal{S}_s) - E(W|\mathcal{M})\Big|(\mathcal{B}_s^+ \vee \mathcal{M})\Big] =$$
$$= E\Big[E(W|\mathcal{M} \vee \mathcal{B}_s^+ \vee \mathcal{S}_s)|(\mathcal{B}_s^+ \vee \mathcal{M})\Big] - E\Big[E(W|\mathcal{M} \vee \mathcal{B}_s^+)|(\mathcal{B}_s^+ \vee \mathcal{M})\Big] =$$
$$= E(W|\mathcal{B}_s^+ \vee \mathcal{M}) - E(W|\mathcal{B}_s^+ \vee \mathcal{M}) = 0 \text{ a.s.}$$

We observe now that $\{E(W|\mathcal{M} \vee \mathcal{S}_s) - E(W|\mathcal{M})\}$ is a random variable $(\mathcal{M} \vee \mathcal{S}_s)$-measurable and, being, by hypothesis, $\mathcal{S}_s \prec\prec \mathcal{B}_s^+|\mathcal{M}$, we get

$$E(W|\mathcal{M} \vee \mathcal{S}_s) = E(W|\mathcal{M}) \text{ a.s.} \quad \forall W \ \Sigma_s - \text{measurable}$$

i.e.

$$\Sigma_s \underline{\|} \mathcal{S}_s | \mathcal{M}$$

$\Diamond$

The theorem 3 of Basu (1959) states that to give the whole sample is the same as to give an ancillary statistic and a complete sufficient statistic, then this ancillary statistic is maximal ancillary. The following theorem represents a predictive version of Basu's result.

Theorem 36 *Let Σ_s a predictive ancillary σ-field and $\mathcal{S}_s$ a complete predictive sufficient σ-field conditionally on Σ_s. Furthermore suppose $\mathcal{S}_s \vee \Sigma_s = \mathcal{B}_s^-$. Then Σ_s is maximal predictive ancillary.*

Proof

Suppose that $\mathcal{G}$ is a predictive ancillary σ-field, such that $\Sigma_s \subseteq \mathcal{G}$. Then it easy to prove that $\mathcal{G}$ is predictive ancillary conditionally on Σ_s (see Mouchart and Rolin, 1984; Corollary 2.6 (ii)). From the Theorem 35 we have

$$\mathcal{S}_s \underline{\|} \mathcal{G} | \Sigma_s$$

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i.e.

$$E(X|\Sigma_s) = E(X|\Sigma_s \vee \mathcal{S}_s) \quad \text{a.s.} \quad \forall X \ \mathcal{G}\text{-measurable}$$

Since $\mathcal{S}_s \vee \Sigma_s = \mathcal{B}_s^-$ we have

$$E(X|\Sigma_s) = E(X|\mathcal{B}_s^-) \quad \text{a.s.} \quad \forall X \ \mathcal{G}\text{-measurable} \tag{15}$$

By definition of ancillarity $\mathcal{G} \subseteq \mathcal{B}_s^-$. Therefore (15) becomes

$$E(X|\Sigma_s) = X \quad \text{a.s.} \quad \forall X \ \mathcal{G}\text{-measurable} \tag{16}$$

(16) implies $\mathcal{G} \subseteq \Sigma_s$. Therefore Σ_s is maximal predictive ancillary.$\Diamond$

Theorem 37 *Let Σ_s and $\mathcal{S}_s$ two independent σ-field, such that $\mathcal{S}_s \vee \Sigma_s = \mathcal{B}_s^-$. Then Σ_s is maximal among σ-field independent of $\mathcal{S}_s$, contained in $\mathcal{B}_s^-$.*

Proof

Let $\mathcal{G}$ a σ-field independent of $\mathcal{S}_s$ such that $\Sigma_s \subseteq \mathcal{G}$. Then $\mathcal{G}\|\mathcal{S}_s|\Sigma_s$ (see Mouchart and Rolin (1984); Corollary 2.6 (ii)), i.e.

$$E(X|\Sigma_s) = E(X|\Sigma_s \vee \mathcal{S}_s) \quad \text{a.s.} \quad \forall X \ \mathcal{G}\text{-measurable}$$

Since $\mathcal{S}_s \vee \Sigma_s = \mathcal{B}_s^-$ e $\mathcal{G} \subseteq \mathcal{B}_s^-$ we have

$$E(X|\Sigma_s) = E(X|\mathcal{B}_s^-) = X \quad \text{a.s.} \quad \forall X \ \mathcal{G}\text{-measurable} \tag{17}$$

From (17) it follows that $\mathcal{G} \subseteq \Sigma_s$. Therefore Σ_s is maximal.$\Diamond$

Corollary 38 *Let Σ_s a predictive ancillary σ-field and $\mathcal{S}_s$ a minimal predictive sufficient σ-field, but not necessarily complete. Suppose $\mathcal{S}_s \vee \Sigma_s = \mathcal{B}_s^-$. Then Σ_s is maximal predictive ancillary among σ-field independent of $\mathcal{S}_s$.*

References

Basu, D. (1955). On statistics independent of a complete sufficient statistics. *Sankhya*, vol. 15, Series A, pp. 377-380.

Basu, D. (1958). On statistics independent of a sufficient statistics. *Sankhya*, vol. 20, Series A, pp. 223-226.

Basu, D. (1959). The family of ancillary statistics. *Sankhya*, vol. 21, Series A, pp. 247-256.

Basu, D. and B. Pereira (1983). Conditional independence in statistics. *Sankhya*, vol. 45., Series A, pp. 324-337.

Campanino, M. and F. Spizzichino (1981). Prediction, sufficiency and representation of infinite sequences of exchangeable random variables. *Technical report.*, Ist. di Matematica G. Castelnuovo Univ. Roma I.

Cifarelli, D.M. and E. Regazzini (1980). Sul ruolo dei riassunti esaustivi ai fini della previsione in contesto bayesiano. Parte I. *Rivista di Mat. per le Scienze Econ. e Soc.*, 3: pp.109-125.

Cifarelli, D.M. and E. Regazzini (1981). Sul ruolo dei riassunti esaustivi ai fini della previsione in contesto bayesiano. Parte II. *Rivista di Mat. per le Scienze Econ. e Soc.*, 4: pp.1-9.

Cifarelli, D.M. and E. Regazzini (1982). Some considerations about mathematical statistics teaching methodology suggested by the concept of exchangeability. *Exchangeability in Probability and Statistics*(G. Koch and F. Spizzichino, eds.), pp. 185-205. North-Holland, Amsterdam.

Dawid, A. P. (1979a). Conditional independence in statistical theory. *J. Roy. Statist. Soc. B*, 41, pp. 1-31.

Dawid, A. P. (1979b). Some misleading arguments involving conditional independence. *J. Roy. Statist. Soc. B*, 41, pp.249-252.

Dawid, A. P. (1982). Intersubjective statistical models. *Exchangeability in Probability and Statistics*(G. Koch and F. Spizzichino, eds.), pp. 217-232. North-Holland, Amsterdam.

Dellacherie, C. and P.A. Meyer (1980). *Probabilité e potentiel (B), Théorie des martingales.* Paris: Hermann

Diaconis, P. and D. Freedman (1984). Partial exchangeability and sufficiency. *Statistics: Applications and New Directions*(J.K. Ghosh and J. Roy, eds.), pp. 205-274.

Florens, J.P. and M. Mouchart and J.M. Rolin (1990). *Elements of Bayesian Statistics.* E.J. Taft and Z. Nashed, eds.

Fortini, S., Ladelli, L. and E. Regazzini (1998). Exchangeability, predictive distributions and parametric models. *Technical Report*, C.N.R.-IAMI 98.5

Fortini, S., Ladelli, L. and E. Regazzini (2000). Exchangeability, predictive distributions and parametric models. *Sankhya*, Vol. 62, Series A, pp. 86-109

Lauritzen, S.L. (1984). Extreme point models in statistic (with discussion). *Lecture Notes in Statist.*, 49, Springer, New York.

Lauritzen, S.L. (1988). Extremal families and system of sufficient statistics in statistic (with discussion). *Scandinavian J. Statist.*, 11, pp. 65-91

Kohen, U. and D.L. Thomas (1975). On statistics independent of a sufficient statistics: Basu's lemma. *Amer. Statis.*, 29(2), pp. 40-42.

Lehman, E.L. and H. Scheffe (1950). Completeness, similar regions and unbiased estimation, Part I.*Sankhya*, vol. 10., Series A, pp. 305-340.

Letta, G. (1981). Sulla nozione di riassunto esaustivo, *Rivista di Mat. per le scienze Econ. e Soc.*, vol. 3., pp. 109-125.

Lloyd, C.J. (1988). Applications and extensions of Basu's results on maximal ancillarity. *Austral. J. Statist.*, 30(2), pp. 163-173.

Mckean, H.P. (1963). Brownian motion with a several-dimensional time. *Theory of Probability and its Applications*, Volume VIII, pp. 335-354.

Mouchart, M. and J.M. Rolin (1984). A note on conditional independence with statistical applications. *Statistica*, 4, pp. 553-584.

Mouchart, M. and J.M. Rolin (1986). On the σ-Algebraic Realization Problem.CORE Discussion Paper 8604, Université Catholique de Louvain, Louvain-la-Neuve, Belgium.

Mouchart, M. and J.M. Rolin (1989). On maximal ancillarity. *Statistica*, 49(1).

Muliere, P. and P. Secchi (1992). Exchangeability, predictive sufficiency and Bayesian bootstrap. *J. Ital. Statist.*, 20, pp. 1203-1221.

Secchi P. (1987). Processi aleatori scambiabili e costruzione di modelli statistici. Degree thesis, Dip. Mat. F. Enriques, Univ. degli Studi di Milano.

Skibinski, M. (1967). Adequate Subfields and Sufficiency. *Annals of Mathematical Statistics*, 38, pp. 155-161.

van Putten, C. and J.H. van Schuppen (1985). Invariance properties of the conditional independence relation. *The Annals of Probability*, 13, pp. 934-945.