Inverse Stochastic dominance, inequality measures and Gini index

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Abstract

We investigate the relationship between the inverse stochastic dominance criterion introduced in Muliere and Scarsini (1989) and inequality dominance when the Lorenz curves intersect. We discuss also the positive dependence stochastic order using the Generalized Lorenz curve (Muliere and Petrone, 1992).

1 Introduction

The research of Corrado Gini in statistical methodology on inequality measures and on association between variables is very important. His contribution on concentration measures is largely well known. Gini was able to construct a solid bridge between statistical theory and economic theory. It was in this context that Gini (1909) first proposed the so-called index δ and then in 1914 the concentration ratio R. It is important to say that Gini's scientific production was rarely the result of just a theoretical formal elaboration but rather it arose from the need to solve concrete problems. The results of Gini are still important in international scientific community and many papers both theoretical and applied are based on his contributions.

The close and direct relationship between the Gini index and Lorenz curve has been one of the main reasons for the wide application of the index in empirical analysis. However, from the theoretical point of view the index has been criticized because of its inconsistency with the orders induced by utilitarian evaluation functions. As argued in Yaari (1981,1988) the appropriate framework of the analysis that is consistent with the Gini index is the dual approach when income distributions are evaluated according to weighted averages of incomes ranked in increasing order and weighted according to their positions. The stochastic order induced by Yaari functional are related with the concept of inverse stochastic dominance introduced in Muliere and Scarsini (1989).

The rest of the paper is organized as follows. In Section 2 we discuss the Lorenz curve and the Gini index; in Section 3 we present a sequence of stochastic orders of degree n; in Section

4 we introduce a sequence of inverse stochastic orders of degree n; in Section 5 we present the Lorenz curve, stochastic dominance and inequality measures; in Section 6 we presents transfers with intersecting Lorenz curves; in Section 7 we present the Generalized Lorenz curve and positive monotone dependence.; in Section 8 we make some concluding remarks and some comments .

2 The Lorenz curve and Gini index.

The seminal work of Gini on the measurement of concentration of income distribution has stimulated a large literature on inequality, welfare and poverty measurement and decision theory. The close and direct relationship between the Gini index R, and the Lorenz curve has been one of the main reasons for the wide application of the index in empirical analysis. However, from the theoretical point of view the index has been criticized because of its inconsistency with the orders induced by utilitarian evaluation functions.

Before the definition of the dominance I need to introduce some preliminaries. Let us introduce a random variable X $\in(0,\infty)$ with differential cumulative distribution, $F(x) = P(X \leq x) = \int_0^t dF(t)$ and mean $E(X) = \int_0^\infty x dF(x)$ exist, finite and $\neq 0$.The Lorenz curve is defined in the Cartesian orthogonal plane by the following equation: $L_X(x) = \frac{1}{E(X)} \int_0^x t dF(t)$, $x \geq 0$. Pietra (1915, pag 282) for the first time in the literature, defined analytically the Lorenz curve using only one equation, that is, very synthetically, $L_X(p) = \frac{1}{E(X)} \int_0^p F^{-1}(u)du$, $0 \leq p \leq 1$.$F^{-1}(x) = inf_x \left[xF(x)\right] \geq u$ is the left continuous inverse distribution function or the quantile function corresponding to F It follow that: $E(X) = \int_0^1 F^{-1}(u)du$. Note that $L_X(0) = 0$ and $L_X(1) = 1$ while in the case of equal distribution $L_X(p) = p$. This author contribution is to having connected the Lorenz curve analytic representation and also for having given a new and substantially stimulus to the studies on Lorenz curve and Gini index. Pietra (1915, pag.780) expressed the Gini index R as follows: $R = \frac{A}{maxA} = \frac{1/2 - \int_0^1 L_X(p)dp}{1/2} = 1 - 2\int_0^1 L_X(p)dp = \frac{\triangle_X}{2E(X)}$ where A is the concentration area (the area between the egalitarian line and Lorenz curve) and $\triangle_X$ is the mean difference. In 1911 Gini published a very important paper on mean difference. The mean difference is defined as :$\triangle_X = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mid x - y \mid f(x)f(y)dxdy$ where f(x) is the density function. It is to easy to see that$\triangle_X = 2\int_0^{+\infty} F(x)(1 - F(x)dx$ in which F(x) is the distribution of the r.v. X. The Generalized Lorenz curve is the Lorenz curve scaled up by average income $GL_X(p) = E(X)L_X(p)$.

3 A sequence of stochastic orders of degree n

The relation of *stochastic dominance* is a fundamental concept of decision theory and economics (Marshall, Olkin and Arnold (2010), Shaked and Shanticumar (2007), Muller and Stoyan (2002)).

For a random variable X$\in$ $(0,\infty)$ we construct the stochastic dominance relation in the following way: $F_1(x) = P(X \leq x)$, and $F_n(x) = \int_0^x F_{n-1}(s)ds$ for$x \in R$, n = 2, ..., m+1, as far as the integral exist.

The stochastic dominance relation of order n , n $\in N$, $\geq_n$, is defined as follows :$F \geq_n G$ iff $F_n(x) \leq G_n(x)$.

It easy to see that:$F_n(x) = \frac{1}{(n-1)!}\int_0^x (x-y)^{n-1}dF(x)$, for every $x \geq 0$. The first order and second order dominance previously defined can be encompassed in a unified approach, stochastic dominance, and extended to higher-order dominance.

Example 3.1.

(a) $F \geq_1 G \Longleftrightarrow X \geq_1 Y$ iff $P(X \leq x) \leq P(Y \leq y)$ with at least < for some x

(b) $F \geq_2 G \Longleftrightarrow X \geq_2 Y$ iff $\int_0^x F(x)dx \leq \int_0^x G(y)dy$ with at least < for some x

(c) $F \geq_3 G \Longleftrightarrow X \geq_3 Y$ iff $\int_0^x \int_0^z F(t)dt \leq \int_0^x \int_0^z G(t)dt$with at least < for some x

Von Neumann and Morgenstern in their book (1944) developed the expected utility theory: for every rational decision maker there exists a utility function *u(.)* such that the decision maker prefers outcome X over outcome Y if and only if : $E(u(X) \geq E(u(Y))$.

In practice, however it is almost impossible to elicit the utility function of a decision maker explicit. Additional difficulties arise when there is a group of decision makers with different utility functions who have to come to a consensus. The stochastic dominance relation has an equivalent characterization by utility function.

Relation : $X \geq_1 Y$ means that $E(u(X)) \geq E(u(Y))$ for every non-decreasing utility function u() for which these expected values exist. The second order stochastic dominance relation $X \geq_2 Y$ means that $E(u(X)) \geq E(u(Y))$ for every non decreasing and concave utility function u(.). $X \geq_3 Y$ means that $E(u(X)) \geq E(u(Y)$ for every non decreasing and concave and with third derivative positive.

The order $\geq_n$ discussed before are presented by Fishburn(1980) and Rolsky (1976) in the following theorem.

Theorem 3.1 For all F and G in $\mathcal{F}$ and n ϵN , $F \geq_n G \longleftrightarrow \int u(x)dF(x) \leq \int u(x)dG(x)$ where $u \epsilon V_n$. $V_n$is the set of all strictly increasing function $u : [0,\infty) \rightarrow R]$ whose derivative exist trough order n and alternatives in sign with $u^{(k)} \geq 0$ for k being odd and $u^{(k)} \leq 0$ for being even.

4 A sequence of inverse stochastic orders of degree n

In this section we mention another sequence of orders that is based on iterated integrals . This order is called$\geq_n^{-1}$ is motivated by Muliere ans Scarsini (1989). Extensions are in Maccheroni, Muliere and Zoli (2005). Some others references are Wang and Young (1998), Zoli (2002, 1999), Aaberge (2004), Dentcheva and Ruszczynski (206), De La Cal and Carcano (2010), Andreoli (2013).

Let $\Im$ be the class of distributions functions on $(0, \infty)$. Let F $\epsilon \Im$; we define $F^{-1}(y) = inf\{x : F(y)\} \geq y)$ with $0 \leq y \leq 1$. This is the left-continuous version of the inverse distribution of F.

Denote recursively: $F_1^{-1}(x) = F^{-1}(x)$, $F_n^{-1}(x) = \int_0^x F_{n-1}^{-1}(s)ds$ for n = 2,3,.... for every x $\in$(0,1). Similarly, we define G_n^{-1} for a distribution function G. For any positive integer m, if the distribution functions F and G of the random variables X and Y, satisfy: $F_m^{-1}(x) \leq G_m^{-1}(x)$,for $0 \leq x \leq 1$, then we denote $X \geq_m^{-1} Y$. It is easy to see that : $X \geq_1^{-1} Y \Longleftrightarrow X \geq_1 Y$. Also , if E(X) = E(Y), then, $X \geq_2 Y \Longleftrightarrow X \geq_2^{-1} Y$.

As shown by Muliere and Scarsini (1989), the third order inverse stochastic dominance is not equivalent to the third direct stochastic dominance

Definition. 4.1.

Let F,G $\epsilon \digamma$.

. $F \geq_n G$ if $F(x) \leq_n G(x)$ for every x ϵ R_+ ($\geq_n$ is the stochastic dominance of degree n, n $\in$ N, (n-SD))

. $F \geq_n^{-1} G$ if $F_n^{-1}(x) \geq G_n^{-1}(x)$ for every x $\epsilon[0, 1]$ ($\geq_n^{-1}$is the n-th degree *inverse stochastic* dominance *(n-ISD))*

The sequence of n-SD and n-ISD form a sequence of progressively finer partial order.

The following lemma is easy proved by induction.

Lemma 5.2

We have $F_n^{-1}(x) = \frac{1}{(n-1)!} \int_0^x (x - y)^{n-1} dF^{-1}(y)$, $0 \leq x \leq 1$.

Proof . The assertion is true for n = 1. Assume that is true for a generic n. Then $F_{n+1}^{-1}|(x) = \int_0^x \frac{1}{(n-1)!} \int_0^t (t-y)^{n-1} dF^{-1}(y)dt = \frac{1}{(n-1)!} \int_0^x \frac{(x-y)^n}{n} dF^{-1}(y)$,where the last equality follows from Fubini's Theorem.

Proposition 4.3

Let F,G $\epsilon \digamma$. If $F \geq_n^{-1} G$, then $F \geq_m^{-1} G$ for all m > n.

Proof. If $F_n^{-1}(x) \geq G_n^{-1}(x)$, for every $x \in [0, 1]$, then $\int_0^x F_n^{-1}(s)ds \geq \int_0^x G_n^{-1}(s)ds$ for every $x \in [0, 1]$, this is $F^{-1} \geq_{n+1} G^{-1}$. Proceeding , by induction, we obtain the result.

It is possible to sse that n-SD and n-ISD are equivalent for n = 1, 2. When n=1, the result is trivial; when n = 2 , it can proved by slight generalization of argument due to Atkinson (1970), who proved the equivalence in the case of absolutely continuous distribution functions having the same mean. When $n \geq 3$ the equivalence does not hold anymore.

The next theorem (see Muliere and Scarsini (1989)) will provide necessary conditions for n-ISD. Let $(X_1 \wedge X_2 \wedge ...X_n) = min_{i=1,...n} X_i$

Theorem 4.4 (Muliere and Scarsini (1989))

If $F \geq_n^{-1} G$, then $E(X_1 \wedge X_2 \wedge ... \wedge X_k) \geq E(Y_1 \wedge Y_2 \wedge ... \wedge Y_k)$

for all $k \geq n-1$, where $X_{1,}X_2,...,X_k$ are i.i.d.rv.'s distributed according to F, and $Y_1, Y_{2,...,}Y_k$ are i.i.d.r.v.'s distributed according to G.

Resorting to the stochastic dominance results of Rolsky (1979) and Fishburn (1976, 1980) we can find a class of inequality measures coherent with $\geq_n^{-1}$ for any positive integer n.

Theorem 4.5 (Muliere and Scarsini (1989))

Let M_n be the class of functions $\varphi : [0,1] \to \mathbb{R}$ such that $\phi(x) = -\int_x^1 (s-x)^{n-1} d\tau(s)$

where τ is a positive measure. Then $F \geq_n^{-1} G$ if and only if $\int_0^1 \phi(x) dF^{-1}(x) \leq \int_0^1 \phi(x) dG^{-1}(x)$, for $\forall \phi \in M_n$

The close and direct relationship between the Gini index and Lorenz Curve has been one of the main reasons for the wide application of the index of Gini in empirical analysis. In the next section we need to discuss . The theorem states that the class of inequality measures of the form $\int_0^1 \phi(x) dF^{-1}(x)$ is coherent with $\geq_n^{-1}$.

5 Lorenz curve, stochastic dominance and inequality measures.

An order $\geq_A$ is finer than another partial $\geq_B$ if $F \geq_A G$ implies $F \geq_B G$. ($\geq_A$ orders all the distributions that $\geq_B$ orders). An order is linear if it orders every pairs of distributions. An inequality measures I is a functional of the distribution that induces a linear order $\geq_I$ in this way:$F \geq_I G$ iff $I(F) \leq I(G)$. If I induces an order $\geq_I$, and $\geq_I$ is finer than $\geq_A$, then I is said to be coherent with $\geq_A$. Now we define an order based on Lorenz curve.

Given two r.v. X is said to Lorenz curve dominate Y: $X \geq_L Y$ iff $L_X(p) \leq L_Y(p)$ for all $p \in [0,1]$ with at least $>$ for some p.

In order to compare (in terms of inequality) pairs of distributions that are not ordered by Lorenz ordering, it is wise to choose (partial or linear) orders that are finer that Lorenz ordering, and therefore to choose inequality measures that are coherent with $\geq_L$. Obviously, E(X) = E(Y), then $F \geq_2 G \Longleftrightarrow F \geq_2^{-1} G \Longleftrightarrow X \geq_L Y$.

The sequence of n-th degree stochastic dominance (n-SD) is a sequence of progressively finer partial orders. Therefore, when the means of the distributions are equal, they are finer then the Lorenz ordering (for n $\geq$ 2).

Example 5.1. (Gini index)

One of the most common measures of inequality is the Gini index R, which is defined as: $R(F) = 1 - 2\int_0^1 L_X(p)dp = 1 - \frac{1}{E(X)}\int_0^1\int_0^p F^{-1} dt dp = 1 - \frac{1}{E(X)} F_3^{-1}(1) = 1 - \frac{E(X_1 \Delta X_{2)}}{E(X)}$

Therefore, whenever $F_X \geq_3^{-1} F_Y$, with E(X) = E(Y), then $R(F_X) \leq R(F_Y)$. This means that Gini index is coherent with third degree inverse stochastic dominance when the r.v's have the same expectation. It is known that the Gini index is coherent with the second stochastic dominance. Our result is stronger, even when the Lorenz curve intersect provided $\geq_3^{-1}$.

Example 5.2. (Donaldson , Weymark)

The indices proposed by Donaldson and Weymark (1980, 1983) (called S-Ginis) generalized the Gini index . For each $k \in N$, they defined an absolute index $Z_k(F_X) = -\int_0^\infty xd(1-F_X(x))^k$ and a relative $k \geq 1$ $I_k(F_X) = 1 - \frac{\int_0^\infty xd(1-F_X(x))^k}{E(X)}$. The index $Z_k(F_X)$ (and therefore $I_k(F_X)$) exist whenever E(X) $< \infty$. Integrating by parts, we obtain $Z_k(F_X) = \int_0^\infty (1-F_X(x))^k dx = E(X_1 \wedge ... X_k)$. Therefore, by Theorem 4.4 , if $F_X \geq_{n+1}^{-1} F_Y$ then $Z_k(F_k) \geq Z_k(F_Y)$ for $k \geq n$. If, furthermore, E(X) = E(Y), then $I_k(F_X) \leq I_k(F_Y)$ for $k \geq n$.It is evident that $I_2 F(_X) = R(F_X)$.

Remark 1

We have introduced a sequence of partial orders $\geq_n^{-1}$ for the distributions which are based on income differentials and are progressively finer, in the sense that , if $n \geq m$ then $\geq_n^{-1}$can order all the pairs of distributions that are ordered by $\geq_m^{-1}$ (and some more) Therefore , as we pass from $\geq_n^{-1}$ to $\geq_{n+1}^{-1}$, none of the previously performed comparisons is demand some others are added.

When $n = 1$, the maximum incomes of the $(100\alpha)\%$ $(0 \leq \alpha \leq 1)$poorest parts of the populations are compared.

When $n = 2$, and integration is performed , and the accumulated incomes of the $(100\alpha)\%$ $0 \leq \alpha \leq 1)$poorest parts the populations are compared.

When $n = 3$, a further integration is performed , and so on .

As n increases, the importance of the lower incomes is stressed and more .

6 Transfers with intersecting Lorenz curves.

When comparing two distributions of well-being, it is of interest to investigate and interpret the transformations by which one distribution is obtained from the other. In our paper we have discussed the transfers with respect the idea of intersecting Lorenz curve and we introduced the inverse stochastic dominance. If the two Lorenz curves intersect, then neither distribution is obtained from the other by a pure series of either progressive or regressive transfers. Either distribution can, however be obtained from the other by a combination of progressive and regressive transfer. In order to understand the role of transfers when the Lorenz curves intersect, we need notions of transfers the combine regressive and progressive transfers in certain ways. See also Mosler and Muliere (1996) for some comments.

The third degree stochastic dominance criteria impose the normative requirement, often considered desirable, of transfers sensitivity.. A Mean Variance preserving transformation (MVPT) is a combination of a regressive transfers and a progressive transfers with the following properties: (i) the regressive transfers occurs at lower income than the progressive transfer does; (ii) the overall effect is to have the variance unchanged.

The same transformation is referred to a favorable composite transfer .

Proposition 6.2.(Shorrocks and Foster (1987))

Let X and Y be two income distributions with equal arithmetic means. The following statements are equivalent:

(i) X is obtained from Y by a finite sequence of progressive transfers and /or MVPT;

(ii) $X \geq_3 Y$

Next for characterizing 3-ISD we consider a particular class of transfers that combine a progressive and regressive transfer. Let (i) the progressive a transfer take place lower down in the distribution (ii) the Gini index remain constant with transfer. This is named a favorable composite positional transfer (FCPT) by Zoli (1999). Like an MVPT, an FCPT combines a progressive transfer and a regressive transfer, the progressive transfer taking place lower down in the distribution.

*Proposition 6.3.(*Zoli (2002))

Let X and Y be two income distributions with equal arithmetic means. The following statements are equivalent.

(i) X is obtained from Y be a finite sequence of progressive transfers only/or FCPT

(ii) $X \geq_3^{-1} Y$

7 Generalized Lorenz curves and positive dependence orders

Muliere and Petrone (1992) introduced the idea to measure monotone dependence in drawing a comparison between the Generalized Lorenz curve of E(Y/X) and Lorenz curve of Y. Roughly speaking, the more is the generalized Lorenz curve of E(Y/X) similar to (or far from) the Lorenz curve of Y, the stronger is the positive (negative) dependence of Y on X. We examine some properties of the Generalized Lorenz curve of $m(X) = E(Y/X)$. The random vector (X,Y) have the distribution function F and let F_1 and F_2 denote, respectively, the marginal distributions of X and Y. We assume that the regression function $m(x) = E(Y/X = x)$ is continuous with finite first derivative m'(x).

Definition.7.1

The generalized Lorenz curve of $m(X) = E(Y/X)$ is defined as:

$L_{E(Y/X)}(p) = \frac{1}{E(E(Y/X))} \int_0^{xp} m(t)F_X(t) = \frac{1}{E(Y)} \int_0^{p} m(F_X^{-1}(z))dz$, $0 \leq p \leq 1$.

In what follow $\mathcal{F}(X,Y)$ denotes the family of all bivariate r.v.'s (X,Y) with monotone dependence, corresponding to positive and negative dependence.

Theorem 7.2

For each (X,Y) and (X',Y') belonging to $(X,Y) \leq_{F^+} (X^+, Y^+)$ if $F_{X=}F_{X'}$ and $F_Y = F_{Y'}$ and $L_{E(Y/X)} \geq [\leq] \, L_{E(Y^+/X^+)}$ for all p $\in$(0,1).

For random vectors (X,Y) and (X',Y') with distribution function in $\mathcal{F}(F_1,F_2)$ the condition $E(Y/X > x) \leq E(Y^+/X^+ > x)$ for every x, can be used to define a positive dependence stochastic order.

We see that if $(X,Y) \leq_{PQD} (X'Y')$ then the $E(Y/X > x) \leq E(Y^+X^+ > x)$ holds (PQD: positive quadrant dependence).

8 Conclusion.

In many empirical works on inequality comparisons, the Gini coefficient is used. It is well known that the Gini coefficient is inconsistent with an utilitarian approach . The results proved in this paper provide an ethical justification for one particular use of the Gini index. We have defined a sequence of progressively finer partial dominance, called n-th degree inverse stochastic dominance . We have showed that Gini index is coherent with third degree inverse stochastic dominance, and not only with second degree inverse stochastic dominance.

Later we have introduced the monotone dependence using a Generalized Lorenz curve.

Corrado Gini was able to discuss problems from different angles thanks to his extensive and profound knowledge of various fields such as Economics, Mathematical Statistics, Biology, Sociology, Demography, Medicine, Informatics and soon. The contributions of Gini relate mainly to the analysis and comparisons of statistical indices for the study of a population rather than a sample. Gini investigated mean values, variability, concentration and the association of random variables. The results of Gini are still important in international scientific community and many papers both theoretical and applied are based on his contributions.

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