Probability Surveys
Vol. 8 (2011) 127–154
ISSN: 1549-5787
DOI: 10.1214/11-PS177

Reviewing alternative characterizations of Meixner process*

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Abstract: Based on the first author's recent PhD thesis entitled "Profiling processes of Meixner type", [50] a review of the main characteristics and characterizations of such particular Lévy processes is extracted, emphasizing the motivations for their introduction in literature as reliable financial models. An insight on orthogonal polynomials is also provided, together with an alternative path for defining the same processes. Also, an attempt of simulation of their trajectories is introduced by means of an original R simulation routine.

AMS 2000 subject classifications: Primary 60G51, 60G07, 60G05; secondary 05E35.
Keywords and phrases: Lévy processes, Meixner process, Esscher transform, equivalent martingale measure, orthogonal polynomials, Meixner-Pollaczeck polynomials, subordinator.

Received March 2010.

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*This is an original survey paper.

1. Introduction: background and motivation

Gaussianity of asset returns has played a central role in pricing theory of financial derivatives: normality of underlying distribution has also been augmented with the assumption of continuity of trajectories when Samuelson introduced in 1965 the geometric Brownian motion [58], subsequently used in the first papers by Black, Scholes and Merton.

The common roots of these works date back in 1900 with the thesis of Bachelier [6], while the physical phenomenon of Brownian motion, usually attributed to Brown [17], was explained by Einstein in [27], also establishing a milestone in the atomistic world view of physics. Bachelier's work is generally considered as both the birth of time-continuous stochastic processes in probability on one hand (Feller in [31] renames Brownian motion as the "Wiener-Bachelier process"), and of the strategies in continuous time to hedge risk in finance on the other. In terms of mathematics, this thesis will deeply influence the researches of Kolmogorov in the 1920's and Itô in the 1950's, until it was rediscovered by Black, Scholes and Merton in 1973 [13]. Modeling asset return prices by means of Black and Scholes model has been a standard procedure in mathematical finance since then.

But, as documented in a large number of papers written by both academics and practitioners, both hypotheses of normality and continuity are contradicted by the data in several pieces of evidence.

There are in fact several points in literature in which this drawback can be spotted; to give a general example, return distributions are more leptokurtic than the Gaussian, as noted by Fama as early as 1963 [30]; this feature is more accentuated when the holding period becomes shorter, and clearly appears in high frequency data. Option prices also exhibit the so called "volatility smile" (see for instance the work of Björk [12] for details) and prices higher than predicted by the Black and Scholes formula for short-dated options. At the same time, under the hypothesis of inability to trade continuously, jumps may be clearly identified also in equity data.

It has to be noticed that this particular hypothesis of uncontinuous trading times is becoming less and less incumbent in present times with the introduction of trading techniques as high-frequency and hyper-frequency trading. Nonetheless, it was a crucial standing point from which financial models based on Lévy processes were introduced.

A complete qualitative description of empirical properties of asset returns can be found for instance in the work by Cont [22].

The introduction and developement of Lévy processes in applications, therefore, finds its reason in the desire to solve problems that Brownian motion based processes weren't able to fit, trying to avoid the basic structural problem described above.

According to this framework, models assuming discontinuities and denying Gaussianity were conceived for mathematical finance.

The presentation of hyperbolic distributions is given by Barndorff-Nielsen in [8], as models for the size of sand grains following an heuristic conjecture by Bagnold, that will be published later in 1980 [7]; soon these distributions entered the field of finance as a starting point for constructing suitable Lévy processes (see for instance also Eberlein and Keller, [26], Eberlein, [25] and related references).

Following this intuition, Meixner process was embedded in financial context by Grigelionis in 1999, [36].

His definition was basically given in "classical" terms of characteristic function and infinite divisibility of the corresponding distribution of the increments. Schoutens proposed a financial application of Meixner process in his technical report of 2002 [62].

Despite their utility, Lévy processes in finance raise other structural problems, some of which still cannot be solved, in particular for specific families of Lévy processes among which Meixner process.

One of the main issues originates from the incompleteness of the markets which they are associated to; it means that there exist an infinite number of martingale measures equivalent to the physical measure describing the underlying price evolution. The problem is that each of them corresponds to a set of derivatives prices which are compatible with the no arbitrage requirement; thus derivative prices are not determined by no arbitrage, but depend on investors'preferences and so the reference to a particular utility function, different from an investor to another, will be needed.

It has been shown by Bellini and Frittelli in [10] that, in general, maximizing utility functions is equivalent to minimizing some kind of distance to the given physical probability measure; in the case of an exponential utility for instance, the dual problem is the minimization of the relative entropy.

A large amount of literature is dedicated to this kind of problems, from the works of Miyahara [52–54], and Fujiwara and Miyahara [34], to the works by Föllmer and Schweizer [32] and Schweizer [65], Hubalek and Sgarra, [39], and the important paper by Kallsen and Shiryaev [42].

All these papers deal with the general problem but never involve Meixner process as an example.

A possible and popular choice for an equivalent martingale measure in the context of exponential Lévy processes is given by the Esscher transform martingale measure, which was given a first formulation in 1932 by actuarial Esscher in [29] and proposed again by Gerber and Shi u in 1994, [35].

Esscher transform, as in the definition in Gerber and Shiu [35], is an useful technique for valuing derivative securities if the logarithms of the prices of the primitive securities are governed by particular stochastic processes with stationary and independent increments. Its mathematical formulation relies on the so called Esscher principle.

Esscher principle is a premium principle, i.e. one of the possible rules for assigning a premium to an insurance risk. It is based on the expectation of the loss under an exponentially transformed distribution properly normalized.

In the already cited paper by Kallsen and Shiryaev, [42], the Esscher martingale measure for exponential processes and the Esscher martingale transform for linear processes (i.e. the representation of the process in terms of stochastic integrals) are introduced to distinguish between two possible kinds of Esscher transforms.

Also, in Grigelionis, [36] the proof of the existence and the explicit evaluation of the parameter which identifies the unique Esscher equivalent martingale measure for Meixner process can be found: this result allows to extend and generalize the usual Black-Scholes formula for a Lévy process different from Brownian motion.

The described approach to the definition of Lévy processes in finance, and in particular of Meixner process, can be classified as a "classical – statistical" one, and as we have described, finds its motivation in the attempt of providing an improvement to the lack of fit of Brownian motion based models.

Literature about Meixner process nonetheless is quite sparse and a more "mathematical" formulation of this model can be found. Its theoretical introduction in this sense dates back to 1998 with a paper by Schoutens and Teugels [64] dealing with orthogonal polynomials and martingales.

In fact the main core of the mathematical theory leading to the generation of Meixner process can be extracted, under suitable conditions on the coefficients, from the solutions of a particular kind of second order differential equation, namely an equation of hypergeometric type. According to the degree and the positivity of the discriminant of the coefficient of the second order term in the starting equation, different families of orthogonal polynomials can be generated. They can be either of continuous variable or discrete variable according to the type (differential or difference) of the generating hypergeometric equation. All these polynomial families fit in a theoretical scheme named the Askey scheme of hypergeometric orthogonal polynomials (see for instance Koekoek and Swarttouw [43] for detailed reference and Lebedev, [44] for classical definitions): it is divided in layers linked by limit transition relations between the different families according to the generalized hypergeometric series which defines the polynomials contained in each family.

For instance, the lowest level is identified by generalized hypergeometric series ${}_2F_0(0)$ (see Schoutens, [61] and also Andrews, Askey and Roy [1] for exact definitions) and contains the Hermite polynomials; climbing one level upwards one finds the ${}_1F_1(1)$ and the ${}_2F_0(1)$ series, defining Laguerre and Charlier polynomials, and so on.

The first three lower layers of Askey scheme contains exclusively polynomial families that enter the group of the so called classical orthogonal polynomials, except for the Meixner-Pollaczeck family, belonging to level ${}_2F_1(2)$ and generating, as we are going to see, the Meixner process.

So another, more theoretical incentive towards the analysis of this process was the attempt to characterize the behavior of the Lévy processes deriving from this particular family of orthogonal polynomials. The same approach was undertaken late in literature, as it is appears from the lack of results regarding Meixner process, mostly in terms of financial theoretical properties and simulation strategies.

Many other mathematical properties have been inspected by Pitman and Yor in [56] for Lévy processes involving hyperbolic functions in their characteristic function (with the Meixner as a particular case); the motivation is that these distributions appear in several problems, especially hitting time problems, distributions of standard Brownian excursions and problems related to random walks and random trees. Often these distributions appear also in analytic number theory.

Always in Pitman and Yor [56], moreover, a hint is made to the problem of expressing these processes in terms of a subordinated Brownian motion, which is solved later by Madan and Yor in [48] by means of suitable subordinators.

Subordinators provide a clear and often practical way to introduce Lévy processes via a Brownian motion with stochastically changed time parameter.

Moreover, because theory of subordinators often allows a useful pattern to obtain simulations of trajectories of Lévy processes, this result is the main reason why we have attempted to simulate trajectories of Meixner process.

At the end of this paper we present simulations of trajectories of a Meixner process (which, by our knowledge, cannot be found anywhere in literature.) obtained via an original R routine based on the theoretical result contained in Madan and Yor [48].

Generally speaking, the more abstract approach to Meixner process makes possible a connection with other fields of the topic as orthogonal polynomials, Stein's method and martingale theory.

A large amount of literature has been produced in terms of mathematical properties of Meixner process. For instance, an analogous of the Brownian motion typical chaotic representation property for a general Lévy process including the Meixner, is shown in a paper by Lytvynov [47], or a possible extension of this process is constructed into a new process called Meixner-type process generated by having the parameters of the characteristic exponent of a standard Meixner process state space dependent by means of suitable pseudo-differential operators, as can be seen in a work by Böttcher and Jacob [14], or again the general analysis of a wide class of Lévy processes (among which the Meixner) by means of their semigroup and theory of integral equations, as in the work of Lev [45].

Moreover an interesting paper by Manstavičius, [49], on topological properties of graphs of Lévy processes, in particular Hausdorff-Besicovitch dimension, assesses this particular quantity to be equal to 1 for every generalized z process and consequently for Meixner process. The result is obtained via the calculation of the Blumenthal - Getoor index β and relating it to Pruitt index, which is in turn almost surely strictly related to Hausdorff-Besicovitch dimension.

The inadequacy of Brownian motion as a backbone of models trying to capture returns of financial assets as opposed to Meixner process, can be shown by easy examples (see for instance Schoutens [62]).

The next part (section 2) of this work recalls the two most famous representations of the characteristic function of an infinitely divisible distribution (see for instance Sato[60] and Bertoin [11] for further details), as reference will be made to them in the following; sections 3 and 4 introduce the basics of Meixner distributions, along with some general estimation techniques for distribution parameters, namely ML estimation and method of moments. Subsection 4.1 of section 4 is devoted to the introduction of the problem of finding a suitable equivalent martingale measure, namely Esscher transform martingale measure.

Section 5 introduces, as announced, a different approach to Lévy processes, and in particular Meixner process, based on orthogonal polynomials, as firstly introduced in the work by Schoutens [61], together with an explicit formula for evaluating Fisher information for Meixner-Pollaczeck polynomials as in Dominici, [24].

Section 6 summarizes the work by Madan and Yor, and describes the theoretical tools to write Meixner process as a suitably subordinated Brownian motion; these results lead to the generation of the simulated trajectories of Meixner process, which are shown at the end of the same section.

The last section 7 collects some possibilities of further research based both on theoretical and applicative issues; the most promising seem to be the investigation of a small deviation problem for Meixner process, and, in the field of applications, the possibility of fitting data coming from seismology, namely the so called “background noise”, by means of Meixner processes.

2. Representations of characteristic functions of infinitely divisible distributions

It is useful at this point to briefly introduce some terminology which will be employed in the following and that has already been cited in the opening section. As references for this section see for instance the works of Sato [60], Bertoin [11], Schoutens [63] and Appelbaum [4].

It is well known that Lévy processes are based on the definition of infinitely divisible (*ID*) distributions, and according to the way the problem of characterizing such distributions was tackled in the 1930’s, two representation formulas due to Khinchine and Kolmogorov were defined for the characteristic function $\phi(\theta)$ of an *ID* distribution.

Definition 1. A distribution of a random variable which for any positive integer n can be represented as a sum of n identically distributed independent random variables is called an infinitely divisible distribution.

Theorem 1 (Lévy-Khinchine canonical representation)**.** *The function $\phi(\theta)$ is the characteristic function of an infinitely divisible distribution if and only if there exist constants $a \in \mathbb{R}$, $\sigma^2 \geq 0$ and a measure ν on $\mathbb{R} \setminus \{0\}$ respecting the condition $\int_{-\infty}^{+\infty} \min\{1, x^2\}\nu(dx) < \infty$ such that*

$$\phi(\theta) = \exp\{\psi(\theta)\}, \tag{2.1}$$

with

$$\psi(\theta) = ia\theta - \frac{\sigma^2}{2}\theta^2 + \int_{-\infty}^{+\infty} (e^{i\theta x} - 1 - i\theta x \mathbb{I}_{\{|x|<1\}})\nu(dx) \tag{2.2}$$

for every θ.

The measure ν is the so called Lévy measure whilst the triplet $(a, \sigma, \nu(dx))$ is the Lévy triplet. The function $\psi(\theta)$ is the characteristic exponent. Constant reference to (2.2) and to the notation introduced in this section will be made, as they represent a typical way to identify and characterize Lévy processes.

In case of an infinite divisible distribution with finite second moment, the following representation formula follows:

Remark 1 (Kolmogorov canonical representation)**.** The function $\phi(\theta)$ is the characteristic function of an infinitely divisible distribution with finite second moment if and only if it can be written as

$$\phi(\theta) = \exp\{\psi(\theta)\}, \tag{2.3}$$

with

$$\psi(\theta) = i\gamma\theta + \int_{-\infty}^{+\infty} \left(e^{i\theta u} - 1 - i\theta u\right) \frac{dK(u)}{u^2}, \tag{2.4}$$

where γ is a real constant, and $K(x)$ is a non decreasing and bounded function such that $K(-\infty) = 0$. The integrand is defined such that for $u = 0$ it is equal to $-\theta^2/2$.

Let us now give the theoretical background of the illustrated example, firstly discussing the approach to Meixner process in standard closed form.

3. From generalized *z* distribution to Meixner distribution

In this section, Meixner distribution will be evaluated in a classical fashion, as a particular case of a more general z distribution. In a paper by Prentice, [57], a class of distributions (z distributions) having the following density

$$f_Z(x) = \frac{2\pi \exp\left\{\frac{2\pi\beta_1}{\alpha}(x-\mu)\right\}}{\alpha B(\beta_1, \beta_2)\left(1 + \exp\left\{\frac{2\pi}{\alpha}(x-\mu)\right\}\right)^{\beta_1+\beta_2}}, \quad \begin{array}{l} x \in \mathbb{R}, \quad \mu \in \mathbb{R}, \\ \alpha > 0, \quad \beta_1 > 0, \quad \beta_2 > 0 \end{array}$$

and $B(\beta_1, \beta_2)$ is the Euler beta function, is introduced.

These distributions were originally introduced in order to simplify the procedure of choice between parametric models: it was in fact observed that for particular values of the parameters, this family contains the exponential, log-normal, Weibull, gamma, generalized gamma, log logistic, χ^2, t and F distributions; hence, discrimination among many usual parametric models is reduced to the choice of a suitable parameter set for the general family. Moreover the comprehensive model shows sufficient robustness to make maximum likelihood results able to be used both for pairwise discrimination and for assessment of the specific models within the more general one.

It is easy to check that the characteristic function of such a density is

$$\phi_Z(u) = \frac{B(\beta_1 + \frac{i\alpha u}{2\pi}, \beta_2 - \frac{i\alpha u}{2\pi})}{B(\beta_1, \beta_2)} \exp(i\mu u), \ u \in \mathbb{R}.$$

With a slight generalization of the preceding case, a probability distribution on $\mathbb{R}$ is called a generalized z distribution ($GZD(\alpha, \beta_1, \beta_2, \delta, \mu)$), see [37], if

$$\phi_{GZ}(u) = \left(\frac{B(\beta_1 + \frac{i\alpha u}{2\pi}, \beta_2 - \frac{i\alpha u}{2\pi})}{B(\beta_1, \beta_2)} \right)^{2\delta} \cdot \exp(i\mu u), \ u \in \mathbb{R}, \delta > 0.$$

With reference to the above terminology, it can be easily shown that $GZD(\alpha, \beta_1, \beta_2, \delta, \mu)$ is infinitely divisible with Lévy triplet $(a, 0, \nu(dx))$, where

$$a = \frac{\alpha\delta}{\pi} \int_0^{2\pi/\alpha} \frac{e^{-\beta_2 s} - e^{-\beta_1 s}}{1 - e^{-s}} \, ds + \mu,$$

$$\nu(dx) = v(x)dx,$$

with

$$v(x) = \begin{cases} \dfrac{2\delta \exp\left\{-\frac{2\pi\beta_2}{\alpha} x\right\}}{x\left(1 - \exp\left\{-\frac{2\pi}{\alpha} x\right\}\right)}, & \text{if } x > 0, \\[2ex] \dfrac{2\delta \exp\left\{-\frac{2\pi\beta_1}{\alpha} x\right\}}{|x|\left(1 - \exp\left\{\frac{2\pi}{\alpha} x\right\}\right)}, & \text{if } x < 0, \end{cases} \quad (3.1)$$

These generalized distributions are completely characterized: in fact let now $\{\kappa_n\}_{n\geq 1}$ be the sequence of the cumulants of $GZD(\alpha, \beta_1, \beta_2, \delta, \mu)$, i.e. the sequence of the coefficients of Taylor's expansion of function defined as $g_{GZ}(u) = \log \phi_{GZ}(u)$; let also $\gamma_1 = \kappa_3/\kappa_2^{3/2}$ be the skewness, and $\gamma_2 = \kappa_4/\kappa_2^2$ the kurtosis; moreover define

$$\nu_n(\beta_1, \beta_2) = \int_0^\infty s^{n-1} \frac{e^{-\beta_2 s} + (-1)^n e^{-\beta_1 s}}{1 - e^{-s}} \, ds, \ n \geq 1$$

then the following formulae hold

$$\kappa_1 = \frac{\alpha\delta}{\pi} \nu_1(\beta_1, \beta_2) + \mu, \qquad \kappa_n = \frac{2\alpha^n \delta}{(2\pi)^n} \nu_n(\beta_1, \beta_2), \ n \geq 2; \quad (3.2)$$

$$\gamma_1 = \frac{\nu_3(\beta_1, \beta_2)}{\left(2\delta\nu_2^3(\beta_1, \beta_2)\right)^{1/2}}, \qquad \gamma_2 = \frac{\nu_4(\beta_1, \beta_2)}{2\delta\nu_2^2(\beta_1, \beta_2)}$$

It can be easily noticed how the results in (3.2) can be considered as a general case for the sequence of the cumulants of Prentice's original distribution, which occurs for $\delta = 1/2$.

A particular case of GZD distribution takes place when

$$\beta_1 = \frac{1}{2} + \frac{\beta}{2\pi}, \qquad \beta_2 = \frac{1}{2} - \frac{\beta}{2\pi},$$

giving place to Meixner distribution $MD(\alpha, \beta, \delta, \mu)$, which we will see in the following.

Definition 2. For all $\alpha > 0, -\pi < \beta < \pi, \delta > 0,$ and $\mu \in \mathbb{R}$

$$MD(\alpha, \beta, \delta, \mu) = GZD\left(\alpha, \frac{1}{2} + \frac{\beta}{2\pi}, \frac{1}{2} - \frac{\beta}{2\pi}, \delta, \mu\right).$$

3.1. Meixner distribution

The density of a random variable X enjoying a Meixner distribution $MD(\alpha, \beta, \delta, \mu)$ is given by

$$f_M(x; \alpha, \beta, \delta, \mu) = \frac{(2\cos(\beta/2))^{2\delta}}{2\alpha\pi\Gamma(2\delta)} \exp\left\{\frac{\beta(x-\mu)}{\alpha}\right\} \left|\Gamma\left(\delta + \frac{i(x-\mu)}{\alpha}\right)\right|^2, \quad (3.3)$$

with $\alpha > 0,\ \beta \in (-\pi, \pi),\ \delta > 0,\ \mu \in \mathbb{R}$ and $\Gamma(\cdot)$ the Euler Gamma function.

μ is a simple location parameter, while α and δ have influence on the peakedness of the distribution; β is a shape parameter influencing primarily the skewness of the distribution.

It is easy to see via a simple standardization, that if $X \sim MD(\alpha, \beta, \delta, \mu)$, then the variable $Z = (X - \mu)/\alpha$ enjoys a $MD(1, \beta, \delta, 0)$.

The characteristic function of X is

$$\phi_{MD}(u) = E\left[e^{iuX}\right] = \left(\frac{\cos(\beta/2)}{\cosh\frac{\alpha u - i\beta}{2}}\right)^{2\delta} \cdot \exp(i\mu u), \quad (3.4)$$

and the cumulant function of X is

$$g_{MD}(u) := \log\phi_{MD}(u) = 2\delta\left[\log(\cos(\beta/2)) - \log\left(\cosh\frac{\alpha u - i\beta}{2}\right)\right] + i\mu \quad (3.5)$$

Let us now discuss the main properties of Meixner distributions:

Property 1. *$MD(\alpha, \beta, \delta, \mu)$ is infinitely divisible with Lévy triplet $(a, 0, \nu(dx))$ with:*

$$a = \alpha\delta\tan\frac{\beta}{2} - 2\delta\int_1^{+\infty}\frac{\sinh(\beta x/\alpha)}{\sinh(\pi x/\alpha)}\,dx + \mu,$$

$$\nu(dx) = \delta\frac{\exp(\beta x/\alpha)}{x\sinh(\pi x/\alpha)}\,dx \quad (3.6)$$

Proof. see Schoutens [63], pgg. 44–45. □

This allows the construction in classical terms of the Meixner process. A consequence of infinite divisibility is that

$$\phi_{MD}(u; \alpha, \beta, \delta, \mu) = [\phi_{MD}(u; \alpha, \beta, \delta/n, \mu/n)]^n,$$

for every $n \in \mathbb{N}$. From this immediately follows that

Property 2. *If $X_j \sim MD(\alpha, \beta, \delta_j, \mu_j)$, $j = 1, \ldots, n$ and they are mutually independent, then*

$$X_1 + \cdots + X_n \sim MD\left(\alpha, \beta, \sum_{j=1}^{n} \delta_j, \sum_{j=1}^{n} \mu_j\right).$$

From the form of the Lévy measure in (3.6) holds the following

Property 3. *$MD(\alpha, \beta, \delta, \mu)$ is self decomposable and has semiheavy tails* (refer for instance to [60] for the definition of self decomposability and semiheavy tails).

Proof. see Grigelionis [36]. □

Property 3 can be also shown in general for *GZD* by means of theorem 5.11.2 in Lukacs [46] observing the form of the Lévy measure in (3.1).

The introduced properties reflect in the corresponding features that we are going to enumerate in section 4 for Meixner process.

Following the same idea given for the generalized z distributions, it is easy to obtain the forms of the first moments of an $MD(\alpha, \beta, \delta, \mu)$;with the same notation, in fact, it holds that

$$\kappa_1 = \alpha\delta \tan\left(\frac{\beta}{2}\right) + \mu, \qquad \kappa_2 = \frac{\alpha^2\delta}{1 + \cos\beta},$$

$$\gamma_1 = \sin\left(\frac{\beta}{2}\right)\sqrt{\frac{2}{\delta}}, \qquad \gamma_2 = 3 + \frac{2 - \cos\beta}{\delta}.$$

3.2. Estimation for the Meixner distribution

Literature on Meixner process usually adopts method of moments as the main parameter estimation method, clearly due to the relative simplicity of computations involved. With slightly more complex calculations, relying on standard likelihood theory, maximum likelihood estimation is also possible for the parameters in this case.

More complex estimation methods either in terms of the whole process or in terms of nonparametric adaptive estimation of the Lévy measure for pure jump Lévy processes (although not directly referring to Meixner processes) are provided for instance in works by Woerner, [66] and Genon-Catalot, [20, 21].

3.2.1. Method of moments estimation

Suppose $x_1, \dots, x_n$ is a random sample drawn from $X \sim MD(\alpha, \beta, \delta, \mu)$; it is relatively simple to estimate the moments of a Meixner distribution by method of moments. Let $\bar{x}$ and s^2 denote as usual the sample mean and uncorrected variance respectively; moreover, defining $\bar{\mu}_k = n^{-1} \sum_{i=1}^n (x_i - \bar{x})^k$, for $k = 2, 3, 4$, let the sample skewness and kurtosis be $\bar{\gamma}_1 = \bar{\mu}_3 / \bar{\mu}_2^{3/2}$, and $\bar{\gamma}_2 = \bar{\mu}_4 / \bar{\mu}_2^2$. Then the usual method of moments procedure leads to these relations

$$\bar{\delta} = \frac{1}{\bar{\gamma}_2 - \bar{\gamma}_1 - 3}, \qquad \bar{\beta} = \frac{\operatorname{sgn}(\bar{\gamma}_1)}{\cos(2 - \bar{\delta}(\bar{\gamma}_2 - 3))},$$
$$\bar{\alpha} = s\sqrt{\frac{\cos\bar{\beta} + 1}{\bar{\alpha}}}, \qquad \bar{\mu} = \bar{x} - \bar{\alpha}\bar{\delta}\tan\left(\frac{\bar{\beta}}{2}\right).$$

Observe that moment estimates do not exist when $\bar{\gamma}_2 < 2\bar{\gamma}_1^2 + 3$.

3.2.2. Maximum Likelihood estimation

Let $x_1, \dots, x_n$ be a random sample as above; the loglikelihood function is given by the expression

$$l_n(\alpha, \beta, \delta, \mu) = \delta \log(2\cos(\beta/2)) - \log(2\alpha\pi) - \log(\Gamma(2\delta)) + \beta\bar{z} + \frac{1}{n}\sum_{i=1}^n \log|\Gamma(\delta + iz_i)|^2,$$

where

$$z_i = \frac{x_i - \mu}{\alpha}, \qquad \bar{z} = \frac{\sum_{i=1}^n z_i}{n}.$$

The MLE $\hat{\theta}_{ML}$ for the vector of parameters $\theta = (\alpha, \beta, \delta, \mu)$ is obtained by usual maximization of

$$\hat{\theta}_{ML} = \arg\max_{\theta \in \Theta} l_n(\theta)$$

with Θ the parameter space for θ. For Meixner distribution it is possible to compute the ML estimate $\hat{I}_n(\hat{\theta}_{ML})$ of the information matrix, since the expressions defining the first two derivatives of loglikelihood functions are explicitly available (for instance see [38], Appendix A), and these expressions can be used to maximize very efficiently the loglikelihood function via Newton-type algorithm based on moments estimates as starting points. Let us now define Meixner process in a standard classical way.

4. Definition of Meixner process

In this section, Meixner process is constructed starting from the definition of Meixner distribution (definition 2) and from its infinite divisibility (property 1). In fact given infinite divisibility of $MD(\alpha, \beta, \delta, \mu)$, a Lévy process can be associated with it as it can be easily seen for instance in Appelbaum [4], Bertoin [11] and Sato [60], which is called the Meixner process.

More precisely, a Meixner process $X = (X_t, t \geq 0)$ is a Lévy process such that

$$X_t \sim MD(\alpha, \beta, \delta t, \mu). \tag{4.1}$$

A standard notation which will be adopted sometimes from now on is $X = MP(\alpha, \beta, \delta, \mu)$.

Standard definition of Meixner process easily allows to collect some basic properties of the process; in fact, summarizing the features of the process, mostly deriving from the corresponding properties of Meixner distribution, we have that this particular Lévy process

- has no Brownian component in fact by property 1, the null value of the middle element of the Lévy triplet is clearly observable;
- its Lévy measure $\nu(dx)$ is characterized by equation (3.6);
- has moments of all orders, as a consequence of the corresponding feature of the generalized z distribution;
- is of infinite variation, in fact it can be verified that

$$\int_{-1}^{+1} |x| \nu(dx) = \infty$$

  with $\nu(dx)$ as in (3.6) the Lévy measure associated to Meixner process;
- has semiheavy tails and is self decomposable, as shown with property 3.

In the following, one possible way to tackle the problem of the existence of an infinite number of equivalent martingale measures for Lévy processes (and in particular for Meixner process) is introduced.

4.1. Esscher transform martingale measure for geometric Meixner process

Esscher transform, as introduced at the beginning of this paper, was developed as one of several candidates for asset pricing in situations of incomplete markets by Gerber and Shiu in [35], referring to an exponential transform originally introduced by Esscher in [29] to approximate the loss distribution of aggregate claims. For a real-valued univariate random variable X on a probability space $(\Omega, \mathcal{F}, P)$ such that $P(X \neq 0) > 0$, Esscher's problem is to construct a measure P' equivalent to P ($P' \sim P$) such that $E_{P'}[X] = 0$. The idea is the following: define a measure $Q \sim P$ by

$$Q(d\omega) = ce^{-X(\omega)^2} P(d\omega), \qquad \text{with } \omega \in \Omega,$$

where c is the normalizing constant $c = 1/E_P[e^{-X^2}]$; then let $\xi(\theta) = E_Q[e^{\theta X}]$ for $\theta \in \mathbb{R}$, and finally

$$Z_\theta(\omega) = \frac{e^{\theta X(\omega)}}{\xi(\theta)}.$$

The map $x \mapsto e^{\theta x}/\phi(\theta)$ is called the *Esscher transform* with parameter θ.

Now the measures P'_θ are costructed via the

$$P'_\theta(d\omega) = Z_\theta(\omega)Q(d\omega) = \frac{e^{\theta X(\omega)}}{\xi(\theta)}Q(d\omega).$$

Then defining $P' = P'_\theta$, it is easy to verify that $P' \sim P$ and $E_{P'}[X] = 0$.

Esscher transform parameter allows to identify, as explained for instance by Hubalek and Sgarra in [40], a particular measure, namely the Esscher equivalent martingale measure for exponential process, with respect to which the "correct" (i.e. no arbitrage) price of a financial activity is equal to its future discounted expected value. In the original paper by Grigelionis [36] one finds the following fundamental theorem, which allows to establish the martingality of discounted geometric Meixner process with respect to the specific measure defined in the claim, and to construct the analogue of the Black-Scholes' formula for Meixner process:

Theorem 2. *The unique Esscher transform parameter $\theta^\star \in \mathbb{R}$ for which the discounted geometric Meixner process*

$$S_0 \exp\{X_t - rt\}$$

with $t \geq 0, S_0 > 0, r \in \mathbb{R}$ is a martingale, is given by

$$\theta^\star = \frac{2}{\alpha} \arccos \frac{|\sin(\alpha/2)|}{\sqrt{1 + \zeta^2 - 2\zeta\cos(\alpha/2)}} - \frac{\beta}{\alpha}.$$

where $\zeta = \exp{(\mu - r)/2\delta}$, $\theta^\star$ solves the equation

$$\cos\frac{\alpha(\theta^\star + 1) + \beta}{2} = \zeta\cos\frac{\alpha\theta^\star + \beta}{2}.$$

Proof. see Grigelionis [36]. □

As it has been pointed out by Kallsen and Shiryaev in [42], two different Esscher martingale transforms exist for Lévy processes according to the choice of the parameter which defines the measure: one turns the ordinary exponential process into a martingale, and the second one turns into a martingale the stochastic exponential. They have been called the Esscher martingale transform for the exponential process and the Esscher martingale transform for the linear process respectively. An approach to pricing contingent claim in conditions of incomplete market is also given by Chan, [19].

It has been shown by Esche and Schweizer in [28] that for exponential Lévy models the Esscher martingale transform for the linear process is also the minimal entropy martingale measure, i.e. the equivalent martingale measure which minimizes the relative entropy, and that this measure has also the property of preserving the Lévy structure of the model (see Hubalek and Sgarra, [40]).

Properties of minimal entropy martingale measure and its financial meaning are provided in a theoretical paper by Frittelli, [33], while in a work by Fujiwara and Miyahara, [34], conditions are provided for the existence of the minimal entropy martingale measure for a general geometric Lévy process of the form $S_t = S_0 e^{X_t}, t \geq 0$, with X_t enjoying an infinitely divisible distribution. Unfortunately these conditions applied to Meixner process are not easily approachable in terms of the involved calculations. In this context, nonetheless, a theoretical paper by Jeanblanc, Klöppel and Miyahara, [41], proves the existence in general under suitable conditions of a sequence of measures which converges to the minimal entropy martingale measure. The main problem of finding a closed form for such a measure in Meixner process case remains anyway unsolved, while a general solution of the problem can be found also in the book by Cont and Tankov [23].

Let us now follow a different route for defining Meixner process and enter the context of orthogonal polynomials.

5. Definition of Meixner process via orthogonal polynomials

Introduction of Meixner process based on orthogonal polynomials theory takes as a starting point the solution of an hypergeometric difference equation under suitable conditions on coefficients (see Schoutens, [61]). These conditions generate different families of orthogonal polynomials that can be obtained in the framework of the well known Askey scheme (see also [43] for a general version of the Askey scheme). In this context stands as a limit case of both Hahn and dual Hahn polynomials the family of Meixner-Pollaczeck polynomials. These Meixner-Pollaczeck polynomials can be introduced by the expression

$$P_n^{(a)}(x;\phi) = \frac{(2a)_n \exp\{in\phi\}}{n} \, {}_2F_1(-n, a+ix; 2a; 1-e^{-2i\phi}),$$

where $a > 0, 0 < \phi < \pi$, and $(a)_n$ is the *Pochhammer symbol*, defined in terms of the Euler Gamma function as

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n > 0;$$

moreover, ${}_2F_1(-n, a_2; b_1; z)$ is a particular case of the generalized hypergeometric series with the first numerator parameter equal to a negative integer.

Namely

$${}_pF_q(-n, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{j=0}^{n} \frac{(-n)_j \ldots (a_p)_j}{(b_1)_j \ldots (b_q)_j} \frac{z^j}{j!}.$$

The orthogonality relation for these polynomials is given by the following

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{(2\phi-\pi)x} |\Gamma(a+ix)|^2 P_m^{(a)}(x;\phi) P_n^{(a)}(x;\phi)\, dx =$$
$$= \frac{\Gamma(n+2a)}{(2\sin\phi)^{2a} n!} \delta_{m,n}, \quad a > 0,\ 0 < \phi < \pi;$$

From [43], for instance, it can also be observed that well known family of Laguerre polynomials $L_n^a(x)$ can be defined through Meixner-Pollaczeck polynomials with

$$\lim_{\phi\to 0} P_n^{(\frac{1}{2}a+\frac{1}{2})}\left(-\frac{x}{2\phi};\phi\right) = L_n^{(a)}(x)$$

Connection between this family of polynomials and corresponding Lévy processes comes from the definition of a particular set of polynomials by means of a power series generating function.

We are going to summarize in sections 5.1, 5.2 and 5.3 the links between orthogonal polynomials and Lévy processes as described by Schoutens in [61].

5.1. Connection between orthogonal polynomials and Lévy processes

Let $f(t)$ and $g(t)$ be functions for which all the necessary derivatives are defined: it can be shown that the equation

$$f(z)\exp\{xu(z)\} = \sum_{m=0}^{\infty} Q_m(x)\frac{z^m}{m!} \tag{5.1}$$

generates a family of polynomials $\{Q_m(x), m \geq 0\}$ when both functions $u(z)$ and $f(z)$ can be expanded in a formal power series and if it holds that $u(0) = 0$, $u'(0) \neq 0$ and $f(0) \neq 0$.

Polynomials $Q_m(x)$ defined like this are of exact degree m and are called *Sheffer polynomials* and any set of such polynomials is called a *Sheffer set*. If now τ is defined as the inverse function of u, i.e. such that $\tau(u(z)) = z$, then τ also can be expanded formally in a power series with $\tau(0) = 0$ and $\tau'(0) \neq 0$.

Now let an additional parameter $t \geq 0$ into the polynomials defined in (5.1) by replacing $f(z)$ with $f(z)^t$.

Definition 3. A polynomial set $\{Q_m(x,t), m \geq 0, t \geq 0\}$ is called a Lévy-Sheffer system if it is defined by a generating function of the form

$$f(z)^t\exp\{xu(z)\} = \sum_{m=0}^{\infty} Q_m(x,t)\frac{z^m}{m!} \tag{5.2}$$

where

(i) $f(z)$ and $u(z)$ are analytic in a neighborhood of $z = 0$;
(ii) $u(0) = 0, f(0) = 1$ and $u'(0) \neq 0$;
(iii) $1/f(\tau(i\theta))$ is an infinitely divisible characteristic function.

If condition *(iii)* is satisfied there is a Lévy process $\{X_t, t \geq 0\}$ defined by the function

$$\phi(\theta) = \phi_X(\theta) = \frac{1}{f(\tau(i\theta))}. \tag{5.3}$$

It can be observed, by means of the work by Anshelevich [2], that both $Q_n(X_t)$ and $Q_n(X_t, t)$ are martingales. The basic link between the polynomials and the corresponding Lévy processes is the following martingale equality

$$E[Q_m(X_t,t)|X_s] = Q_m(X_s,s), \quad 0 \leq s \leq t, \quad m \geq 0. \tag{5.4}$$

In fact it holds, for the left hand side of the equality above, that

$$\sum_{m=0}^{\infty} E[Q_m(X_t,t)|X_s]\frac{z^m}{m!} = E\left[\sum_{m=0}^{\infty} Q_m(X_t,t)\frac{z^m}{m!}\middle| X_s\right] =$$
$$= E[f(z)^t \exp\{u(z)X_t\}|X_s] =$$
$$= f(z)^t \exp\{u(z)X_s\}E[\exp\{u(z)(X_t - X_s)\}|X_s];$$

whilst for the righthand side of (5.4),

$$\sum_{m=0}^{\infty} Q_m(X_s,s)\frac{z^m}{m!} = f(z)^s \exp\{u(z)X_s\}$$

And then the combination of the two expressions above with the stationary increments property of Lévy process leads to

$$E[\exp\{u(z)(X_t - X_s)\}|X_s] = f(z)^{s-t};$$

comparing this relationship with the equation determining the Lévy process

$$E[\exp\{i\theta(X_t - X_s)\}|X_s] = \phi(\theta)^{t-s},$$

then it can be observed that (5.4) holds true if and only if (5.3) holds.

5.2. Meixner set of orthogonal polynomials

In his 1934 paper [51], Josef Meixner determined all sets of orthogonal polynomials that satisfy relation (5.1).

By application of operator $\tau(D)$ defined by recurrence as

$$\tau(D)Q_m(x) = mQ_{m-1}(x), \quad m \geq 0,$$

where $D = d/dx$ is the differential operator with respect to x, by means of the relation

$$Q_{m+1}(x) = (x + l_{m+1})Q_m(x) + k_{m+1}Q_{m-1}(x), \tag{5.5}$$

where $l_m \in \mathbb{R},\ k_m < 0,\ m \geq 2$, a differential equation for function τ can be obtained:

$$\tau'(y) = 1 - \lambda\tau(y) - \kappa\tau^2(y), \tag{5.6}$$

where $l_{m+1} - l_m = \lambda$ and $\kappa \leq 0$.

Moreover the following differential equation for $f(z)$ can be obtained:

$$\frac{f'(z)}{f(z)} = \frac{k_2 z + l_1}{1 - \lambda z - \kappa z^2}. \tag{5.7}$$

with $k_2 < 0$. The above denominator can now be factorized as:

$$1 - \lambda z - \kappa z^2 = (1 - \alpha z)(1 - \beta z),$$

for two real numbers α, β, with $\alpha\beta > 0$. Considering (5.6) a differential equation for $u(z)$ can be drawn:

$$u'(z) = \frac{1}{(1-\alpha z)(1-\beta z)}. \tag{5.8}$$

Explicit solutions of equations (5.6), (5.7), (5.8) are known. This allows to determine the underlying process via an identification of parameters in Kolmogorov representation (2.4); the following equality holds:

$$\psi(\theta) = \log\phi(\theta) = \begin{cases} \dfrac{i\theta(\alpha+\beta) + \log((\alpha-\beta)/(\alpha e^{i\alpha\theta} - \beta e^{i\beta\theta}))}{\alpha\beta}, & \text{if } 0 \neq \alpha \neq \beta \neq 0, \\ \dfrac{i\theta}{\alpha} - \dfrac{\log(1+i\alpha\theta)}{\alpha^2}, & \text{if } \alpha = \beta \neq 0, \\ \dfrac{i\theta}{\alpha} - \dfrac{(1-\exp(-i\alpha\theta))}{\alpha^2}, & \text{if } \alpha \neq \beta = 0, \\ -\dfrac{\theta^2}{2}, & \text{if } \alpha = \beta = 0. \end{cases}$$

The study of generating functions for polynomials belonging to Meixner class, is carried further with two papers by M.Bożejko and E.Lytvynov [15, 16].

5.3. Meixner process from Meixner-Pollaczeck polynomials

When $\alpha \neq 0, \beta = \bar{\alpha}$, the solutions of (5.6), (5.7), (5.8) above become

$$\begin{aligned} u(z) &= \frac{1}{\alpha - \bar{\alpha}} \log\left(\frac{1-\bar{\alpha}z}{1-\alpha z}\right), \\ f(z) &= (1-\alpha z)^{\frac{1}{\alpha(\alpha-\bar{\alpha})}} (1-\bar{\alpha}z)^{-\frac{1}{\bar{\alpha}}(\alpha-\bar{\alpha})}, \\ \psi(\theta) &= i\frac{\alpha+\bar{\alpha}}{\alpha\bar{\alpha}}\theta + \frac{1}{\alpha\bar{\alpha}} \log\left(\frac{\alpha-\bar{\alpha}}{\alpha\exp(i\alpha\theta) - \bar{\alpha}\exp(i\bar{\alpha}\theta)}\right) \end{aligned}$$

and the following expression is obtained for the basic polynomials

$$\sum_{m=0}^{\infty} Q_m(x;t)\frac{z^m}{m} = (1-\alpha z)^{\frac{t-\alpha x}{\alpha(\alpha-\bar{\alpha})}} (1-\bar{\alpha}z)^{\frac{(\bar{\alpha}x-t)}{\bar{\alpha}(\alpha-\bar{\alpha})}}$$

Since $\beta = \bar{\alpha}$ it is natural to write $\alpha = \rho\exp(i\zeta)$; let now be X_t as in (4.1) with $\mu = 0$ and also $X_1 = X$; it is necessary now to identify function $\psi(\theta)$ above with a suitable variant $\psi_X(\theta)$.

With the introduced expression for α, the argument within the logarithm in the above expression for $\psi(\theta)$ can be rewritten in the form

$$\frac{\alpha-\bar{\alpha}}{\alpha\exp(i\alpha\theta) - \bar{\alpha}\exp(i\bar{\alpha}\theta)} = \exp(-i\theta\rho\cos\zeta)\frac{\sin\zeta}{\sin(\zeta + i\theta\rho\sin\zeta)}. \tag{5.9}$$

Hence we put $\zeta = \pi/2 + a/2$ in the expression for $\psi_X(\theta)$. Taking $\delta = 1$ again in (4.1), it holds that

$$\psi(\theta) = i\frac{\theta}{\rho}\cos\zeta + \frac{1}{2\rho^2}\psi_X(2\rho\theta\sin\zeta). \tag{5.10}$$

Recalling now representation (2.2), and that in presence of an infinitely divisible distribution with characteristic function $\phi(\theta)$, a Lévy process X_t may be defined through the relations

$$\exp(\psi_X(\theta)) = \phi_X(\theta) = E[\exp(i\theta X_1)] = \phi(\theta),$$

it can be observed that if $\{Y_t, t \geq 0\}$ is a Lévy process with characteristic function

$$E[e^{i\theta Y_t}] = \exp\{t\psi_Y(\theta)\},$$

then also $X_t = At + BY_{Ct}$, with $C > 0$ is a Lévy process determined by

$$\psi_X(\theta) = i\theta A + C\psi_Y(B\theta). \tag{5.11}$$

From (5.10), the identification between the processes $\{X_t, t \geq 0\}$ and the newly defined Meixner process $\{H_t, t \geq 0\}$ is then achieved by choosing

$$A = \frac{1}{\rho}\cos\zeta, \quad B = 2\rho\sin\zeta, \quad C = (2\rho^2)^{-1}.$$

So

$$X_t = \frac{t}{\rho}\cos\zeta + 2\rho\sin\zeta H_{t/(2\rho^2)}.$$

The Meixner-Pollaczeck polynomial is defined for $\lambda > 0$ and $0 < \zeta < \pi$ by

$$\sum_{m=0}^{\infty} P_m(y;\lambda,\zeta)\frac{w^m}{m!} = \frac{(1-\exp\{i\zeta\}w)^{-\lambda+iy}}{(1-\exp\{-i\zeta\}w)^{\lambda+iy}}$$

Here the identification is simple and leads to

$$w = z\rho, \quad \lambda = \frac{t}{2\rho^2}, \quad y = \frac{x}{2\rho\sin\zeta} - \frac{t}{2\rho^2}\cot\zeta.$$

Moreover, the equality

$$Q_m(x,t) = m!\rho^m P_m\left(\frac{x}{2\rho\sin\zeta} - \frac{t}{2\rho^2}\cot\zeta,\ \frac{t}{2\rho^2},\ \zeta\right),$$

easily brings to the martingale property

$$E\left[P_m\left(H_{\frac{t}{2\rho^2}};\ \frac{t}{2\rho^2},\ \zeta\right)\middle| H_{\frac{s}{2\rho^2}}\right] = P_m\left(H_{\frac{s}{2\rho^2}};\ \frac{s}{2\rho^2},\ \zeta\right).$$

A consequence is that the Meixner$(1,\ 2\zeta-\pi,\delta,0)$ distribution is the measure of orthogonality of the Meixner-Pollaczeck polynomials $\{P_n(x;\delta,\zeta), n = 0,1,\ldots\}$.

Moreover the monic Meixner-Pollaczeck polynomials $\{\tilde{P}_n(x;\delta,\zeta), n = 0,1,\ldots\}$ are martingales for the Meixner process $(\alpha = 1, \delta = 1, \zeta = (\beta+\pi)/2)$:

$$E\left[\tilde{P}_n\left(H_t;\ t,\ \zeta\right)\middle| H_s\right] = \tilde{P}_m\left(H_s;\ s,\ \zeta\right).$$

It remains to determine K in (2.4); recalling the usual exponential form for α one gets

$$\int_{-\infty}^{+\infty} \exp(i\theta x) dK(x) = \left(\frac{\sin \zeta}{\sin(\zeta + i\theta\rho \sin \zeta)} \right)^2 .$$

A little algebra reveals that K has a derivative with expression

$$\frac{dK(y)}{dy} = \frac{\sin \zeta}{\pi \rho} \left| \Gamma \left(1 - \frac{iy}{2\rho \sin \zeta} \right) \right|^2 \exp\left(-\frac{y(\pi - 2\zeta)}{2\rho \sin \zeta} \right).$$

From a strictly theoretical point of view, following the approach of Dominici in [24] it is possible to evaluate the Fisher information of the Meixner-Pollaczeck orthogonal polynomials, a concept introduced for general orthogonal polynomials by Sanchéz-Ruiz and Dehesa in [59]. They considered a sequence of real polynomials, orthogonal with respect to the weight function $\rho(x)$ on the interval $[a, b]$

$$\int_a^b P_n(x) P_m(x) \rho(d) dx = h_n \delta_{n,m}, \quad m, n = 0, 1, \ldots \tag{5.12}$$

with $\deg(P_n) = n$. Introducing the normalized density functions

$$\rho_n(x) = \frac{[P_n(x)]^2 \rho(x)}{h_n}, \tag{5.13}$$

they in fact defined the Fisher information corresponding to the densities (5.13) by

$$\mathcal{I}(n) = \int_a^b \frac{[\rho_n'(x)]^2}{\rho_n(x)}. \tag{5.14}$$

Applying the last formula to the classical hypergeometric polynomials, in Sanchéz-Ruiz and Dehesa [59] $\mathcal{I}(n)$ for Jacobi, Laguerre and Hermite polynomials is evaluated.

The main theorem in Dominici [24] is now the following:

Theorem 3. *The Fisher information of the Meixner-Pollaczeck polynomials is given by*

$$I_\phi(P_n^{(a)}) = \int_{-\infty}^{+\infty} \left[\frac{\partial}{\partial \theta} \rho_n(x) \right]^2 \frac{1}{\rho_n(x)} \, dx = \frac{2[n^2 + (2n+1)a]}{\sin^2(\phi)}, \; n = 0, 1, \ldots \tag{5.15}$$

Proof. see Dominici [24]. □

As a closing remark, it is useful to recall that orthogonal polynomials also provide an auxiliary link with Stein's method. This is basically a procedure of finding approximations for the distribution of a random variable, which at the same time give an estimation of the approximation error involved. This means finding bounds on the distance between two probability distributions with respect to a specific metric. As again shown in Schoutens [61], a key theoretical element is the Stein-Markov operator; it can be shown that it obeys the same hypergeometrical differential (or difference according to the discreteness of involved variable) equation which generates the different families of orthogonal polynomials mentioned above. An important step of the theory is then to formally solve the Stein-Markov equation by means of orthogonal polynomials, which basically represent the eigenfunctions of the cited operator. For further details, for instance see again Schoutens [61]. Furthermore, the original paper by Meixner, [51], is the basis of more advanced studies on the so called Meixner class of measures; see for instance Anshelevich [3].

As a closing topic, let us inspect the possibility of formulating Meixner processes in terms of a subordinated Brownian motion.

6. Meixner process as a subordinated Brownian Motion

An extensive list of definitions of Lévy processes used in finance can be found for instance in Schoutens [63]. Due to a work by Monroe [55] it is known that any semimartingale can be written as a time changed Brownian motion, and so even Meixner process has theoretically its representation as a time changed Brownian motion.

It turns out, nonetheless, that as we have briefly introduced, for some Lévy processes like the Variance Gamma process and the Normal Inverse Gaussian process, the alternative construction as time changed Brownian motions is completely characterized by their specific time change; other processes such as the CGMY process, defined as the pure jump process having the following Lévy measure

$$\nu_{CGMY}(x) = \begin{cases} C\dfrac{\exp(-G|x|)}{|x|^{1+Y}}, & \text{if } x < 0 \\ C\dfrac{\exp(-Mx)}{x^{1+Y}}, & \text{if } x > 0 \end{cases}$$

with $C, G, M > 0$ and $Y \in (-\infty, 2)$, as described in [18], or the Meixner process, are directly identified by their Lévy measure and their specific time change is not a priori known.

This problem has been solved in a work by Madan and Yor, [48], in which a complete characterization of Meixner process as a time changed Brownian motion can be found. Here follows a brief summary of their idea.

6.1. Lévy measure of a subordinated Brownian motion

Suppose the Lévy process X_t is obtained by subordinating a Brownian motion with drift (i.e. the process $\theta u + W_u$, for $\{W_u,\ u \geq 0\}$ a standard Brownian motion) by an independent subordinator Y_t with Lévy measure $\nu(dy)$. By a result in Sato, [60], (30.8), pg.198, the Lévy measure of the process X_t is given by $\mu(dx)$, where

$$\mu(dx) = \int_0^\infty \frac{1}{\sqrt{2\pi y}} \exp\left\{-\frac{(x-\theta y)^2}{2y}\right\} \nu(dy)\, dx.$$

6.2. Explicit time change for Meixner process

Lévy measure $\nu(dx)$ for the Meixner process has been previously introduced in (3.6). The writing of this process as a time changed Brownian motion is given in the already cited paper by Madan and Yor ([48], from pg.20 on): here it is clarified how it is necessary to search for the Lévy density $l(u)$ of a subordinator such that

$$\nu(dx) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi y}} \exp\left\{-\frac{(x-Ay)^2}{2y}\right\} l(y)\, dy\, dx = \\ = e^{Ax} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi y}} \exp\left\{-\frac{x^2}{2y} - \frac{A^2 y}{2}\right\} l(y)\, dy\, dx.$$

Setting $A = \beta/\alpha$ the following must hold for a suitable $l(y)$:

$$\frac{\delta}{x \sinh\left(\frac{\pi x}{\alpha}\right)} = \int_{0}^{+\infty} \frac{1}{\sqrt{2\pi y}} \exp\left\{-\frac{x^2}{2y} - \frac{A^2 y}{2}\right\} l(y)\, dy$$

With somewhat delicate algebra one obtains

$$l(u) = \frac{\delta\alpha}{\sqrt{2\pi u^3}} g(u),$$

where

$$g(u) = P\left(M_1^{(3)} \geq C\sqrt{u}\right) \exp\left\{-\frac{A^2 u}{2}\right\},$$

with

$$\frac{1}{\left[M_1^{(3)}\right]^2} = T_1^{(3)},$$

and

$$M_1^{(3)} = \max_{t \leq 1} R_t^{(3)}$$

for $R_t^{(3)}$ the $BES(3)$ process.

For the absolute continuity of the subordinator with respect to the one sided stable 1/2 subordinator, it is required, and easily verified that

$$\int \frac{1}{\sqrt{u^3}} \left(\sqrt{g(u)} - 1\right)^2 du < \infty$$

Also, for the simulation of Meixner process as a time changed Brownian motion it is possible to represent (see Pitman and Yor's paper [56])

$$P\left(M_1^{(3)} \geq C\sqrt{u}\right) = \sum_{n=-\infty}^{+\infty} (-1)^n \exp\left\{-\frac{n^2\pi^2}{2C^2 u}\right\}.$$

6.3. Simulation of the Meixner process

The first step is to simulate the jumps of the one sided stable 1/2 subordinator with Lévy density

$$k(x) = \frac{\delta\alpha}{\sqrt{2\pi x^3}}, \quad x > 0.$$

The small jumps of the subordinator are approximated using the drift

$$\zeta = \delta\alpha\sqrt{\frac{2\varepsilon}{\pi}}$$

while the arrival rate for the jumps above ε is

$$\lambda = \delta\alpha\sqrt{\frac{2}{\pi\varepsilon}}$$

and the jump sizes for the one sided stable 1/2 subordinator are

$$y_j = \frac{\varepsilon}{u_j^2}$$

for an independent uniform sequence $\{u_j\}$. Then the function $g(y)$ at the point y_j is evaluated, and the time change variable is defined as

$$\tau = \zeta + \sum_j y_j \mathbb{I}_{\{g(y_j) > w_j\}},$$

for another independent uniform sequence $\{w_j\}$. It can also be observed that the function $g(y)$ only uses the parameters α, β and is independent of the parameter δ.

Finally the value of the Meixner random variable or equivalently the unit time level of the process is then generated as

$$X = \frac{\beta}{\alpha}\tau + \sqrt{\tau}Z, \tag{6.1}$$

where Z is an independent standard normal random variable.

The result of some simulations we have carried out relying on the theoretical algorithm described above, is shown in the following figures. Their importance is given by the possibilty of determine the influence of the parameters in the distribution of the increments of the process: this in turn gives the opportunity to adjust the model and fit it to data via a suitable parameter estimation method.

Here are some examples, obtained with an original R routine, for different values of the parameters:

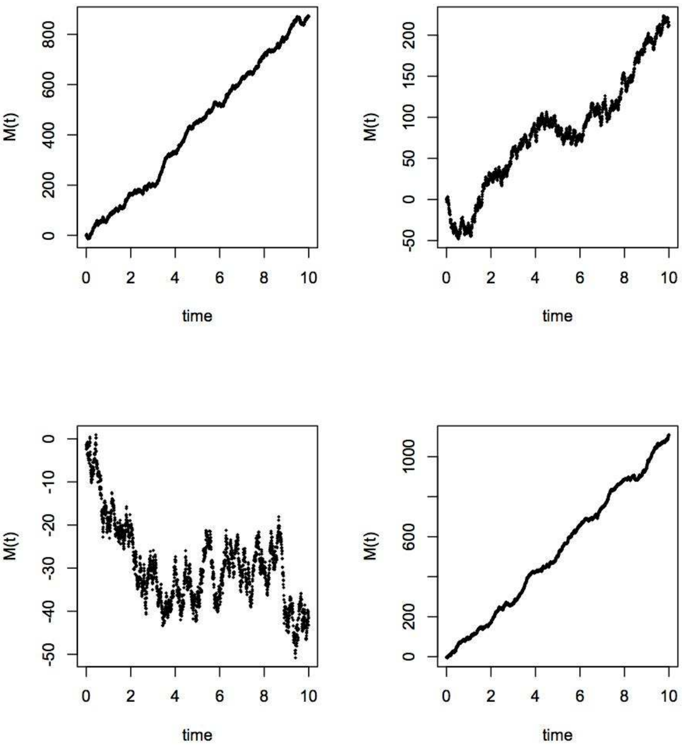

FIG 1. *Possible trajectories of a Meixner process with the following parameter values for the triplet* (α, β, δ)*:* $(0.25, 0.002, 0.2)$ *(left),* $(0.25, 0.002, 20)$ *(right).*

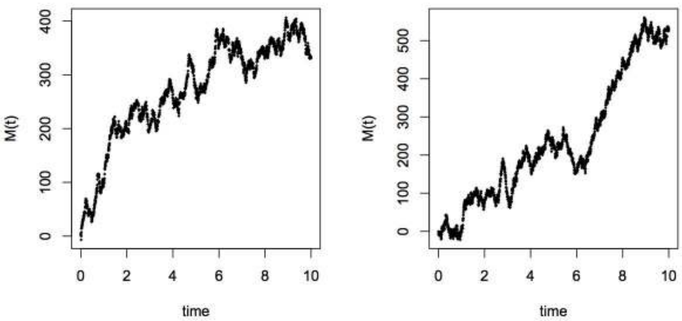

FIG 2. *Possible trajectories of a Meixner process with the following parameter values for the triplet* (α, β, δ)*:* $(0.25, 0.02, 2)$ *(upper left),* $(0.25, 0.002, 2)$ *(upper right),* $(25, 0.002, 2)$ *(lower left),* $(0.25, 0.002, 2)$ *(lower right).*

7. Conclusions and further research ahead

It is quite clear that departure from classical mathematical models for finance governed by Brownian motion and Gaussian distribution, although solving some structural issues, is not completely straightforward. Theoretical properties and general conditions are mostly well known, but one important problem is given by computational effort, for instance for the evaluation of the minimal entropy martingale measure.

General results exist, as in the book by Cont and Tankov [23], on how to construct minimal entropy martingale measure for exponential Lévy models and its relationship with Esscher transform martingale measure. Some calculations have been performed during this work specifically for Meixner process, but have not given any result at the moment. Other obscure points concern minimal martingale measure and optimal variance martingale measure.

In any case, our main goal was to focus interest on a process that undoubtedly gives better performances when considered as a mathematical financial model. Another point is that this claim has been shown not making use of ad-hoc software and creating from scratch all the pieces of software that in literature are often just suggested or hinted at.

So one of the goals was to show that it is possible to apply this process in modeling financial data with small efforts in terms of programming.

Another interesting issue for developing research is given by the study of subordinated Meixner process, which is missing to our knowledge from analyzed literature.

Referring to a work by Aurzada and Dereich, [5], a small deviations problem for Meixner process can be investigated.

From an applied point of view, it has turned out that some seismological graphics happen to be very similar to volalility clusters graphics shown for instance by Schoutens in [62], Fig.6; namely, some background noise recorded on a daily basis by seismometers and having causes depending on human activity and on natural phenomena not strictly of a seismological origin, has such a representation. It can be interesting to investigate the possibility of fitting these data with Meixner-SV models such as as the one introduced by Schoutens.

The main impression is that in this difficult topic many results are added by very little pieces. Still by 2006 no one had been able to express Meixner process as a subordinated Brownian motion, which opened the possibility to simulate the process as we have seen. Nonetheless the study of simulated trajectories as the ones we have introduced here is still missing.

The importance of these models is clear, providing a real flexible and fitting instrument mainly for financial applications; similar models such as hyperbolic models as introduced in [8, 9, 26], have been employed in theoretical quantum physics and in modeling natural phenomena such as turbulence or sand deposits.

We have also shown how these kinds of models could provide a sort of aperture towards different fields of mathematics, involving statistics indirectly, such as the theory of differential equations and orthogonal polynomials.

This gives the models a sort of mathematical reliability, descending from Meixner's cited 1934 work, which was known and settled mainly for Brownian motion only up to that moment.

Acknowledgements

The authors would like to thank the Editor and the reviewers of this paper for their careful and detailed annotations.

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