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CHARACTERIZATIONS OF SOME PROBABILITY DISTRIBUTIONS USING THE PRODUCTS OF INDEPENDENT RANDOM VARIABLES

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Abstract: Suppose W, X, X_1 and X_2 are independent random variables and further suppose that X and $W(X_1 + X_2)$ are identically distributed. We give some characterizations of probability distributions based on such identities.

1. Introduction

Suppose a random variable U is uniformly distributed on the interval [0,1]. Let Y, Y_1 and Y_2 be independent and identically distributed nonnegative random variables independent of the random variable U. Further suppose that

$$Y \stackrel{d}{=} U(Y_1 + Y_2)$$

in the sense that the random variables Y and $U(Y_1 + Y_2)$ have the same distribution. Kotz and Steutel (1988) proved that the above equation characterizes the standard exponential distribution. We now extend such results to other distributions.

2. Preliminaries

Suppose $f(x)$ is a real-valued function that is defined almost everywhere for $x \geq 0$ and is such that

$$\int_0^1 |f(x)| x^{c_1 - 1} dx < \infty \quad \text{and} \quad \int_1^\infty |f(x)| x^{c_2 - 1} dx < \infty$$

for some real numbers c_1 and c_2 with $c_1 < c_2$. Then the Mellin tranform $\hat{f}(s)$ of $f(x)$

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is defined by

$$\hat{f}(s) = \int_0^\infty x^{s-1} f(x) dx$$

where $s = c + i\tau$ is a complex variable with $c_1 < c < c_2$.

If the Mellin transform exists and is an analytic function of the complex variable s for $c_1 \leq Re(s) \leq c_2$, where c_1 and c_2 are real, then the inversion integral converges to the function $f(x)$, that is,

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \hat{f}(s) x^{-s} ds$$

where $c_1 < c < c_2$, the integration path is parallel to the imaginary axis of the complex plane s and the integral is understood in the sense of the Cauchy principal value (cf. Polyanin and Manzhirov (1998), pp. 433-434; Springer (1979), pp.30-31). For the general theory of Mellin transforms, see Paris and Kaminski (2001).

3. Main result

Let X, X_1 and X_2 be nonnegative independent and identically distributed (i.i.d) random variables which are independent of another random variable W. Our problem is to determine the distribution of the random variable W such that

$$X \stackrel{d}{=} W(X_1 + X_2). \tag{1}$$

Let $F_X(.)$ denote the distribution function of a random variable X and $f_X(.)$ denote the probability density function of X whenever it exists. Let

$$\phi_X(u) = E\left(e^{-uX}\right) \equiv \int_R e^{-uX} dF_X(x), u \geq 0 \tag{2}$$

denote the Laplace transform of the distribution of X. Then (1) is equivalent to

$$\phi_X(u) = \int_R \phi_X^2(uw) f_W(w) dw. \tag{3}$$

Then, in order to find $f_W(.)$, we need to solve the integral equation (3).

Theorem 3.1: Suppose that X, X_1 and X_2 are nonnegative independent and identically distributed random variables which are independent of another absolutely continuous random variable W with a probability density function $f_W(w)$. Further suppose of the required solution $f_W(w)$ of the equation (3). Applying now the inversion formula for the Mellin transforms, we obtain the solution of the integral equation (3) in the form given by (4) where $c_1 < c < c_2$.

Remarks 3.1: Obviously we can generalize the above discussion to find the distribution of a random variable W independent of n i.i.d. nonnegative random variables $X_i,\ 1 \leq i \leq n$ and X satisfying the relation that

$$X \overset{d}{=} W\left[X_1 + X_2 + \cdots + X_n\right].$$

In analogy with (3), we obtain the integral equation

$$\phi_X(u) = \int_0^\infty \phi_X^n(uw) f_W(w)dw$$

in this case.

4. Applications to characterizations of some probability distributions

Kotz and Steutel (1988) and Yeo and Milne (1989) have given univariate and multivariate characterizations of the exponential and the uniform distributions based on the product

$$Z = UV$$

of two independent random variables U and V, where

$$V = X + Y$$

is the sum of two independent and identically distributed non-negative random variables X and Y. As formulated by Yeo and Milne (1989), the result of Kotz and Steutel (1988) is that any two of the following conditions imply the third condition:

(i) Z has the same distribution as that of X and Y;

(ii) U has a uniform distribution on the interval $(0,1)$; and

(iii) Z has the standard exponential distribution.

which implies that

$$f_W(w) = \begin{cases} 1, & 0 < w < 1 \\ 0, & \text{otherwise.} \end{cases}$$

We now combine the results obtained above in the following theorem.

Theorem 4.2: Suppose that X, X_1 and X_2 are nonnegative independent and identically distributed random variables which are independent of another absolutely continuous random variable W and suppose that (1) holds. Then any two of the following three conditions imply the third condition:

i) X has the same distribution as that of X_1 and X_2;

ii) W has a uniform distribution on $(0,1)$; and

iii) X has standard exponential distribution.

Proof: We have observed that the conditions (i) and (iii) imply (ii) from Theorem 4.1. It is clear that the conditions (ii) and (iii) imply (i). This can be seen by the following arguments. From the equation (8), we have

$$\hat{\phi^2_X}(s)\hat{f_W}(1-s) = \hat{\phi_X}(s). \tag{9}$$

Since W is uniform on $[0,1]$, the Mellin transform of W is $\hat{f_W}(s) = \frac{1}{s}$. Since the random variable X is standard exponential, the Mellin transform of the Laplace transform of X is

$$\hat{\phi_X}(s) = \frac{\pi}{\sin(\pi s)}.$$

Therefore

$$\hat{\phi^2_X}(s) = \frac{\pi(1-s)}{\sin(\pi s)} = -\frac{\pi(s-1)}{\sin(\pi s)}$$

from the equation (9). Using the inversion formula of Mellin and Laplace transforms, we obtain that the random variables X_1 and X_2 have standard exponential distributions. We now prove that (i), (ii) imply (iii). From (3) and (i), we have

$$\phi_X(u) = \int_R [\phi_X(uw)]^2 f_W(w)dw.$$

Condition (ii) implies that

$$\begin{aligned} \phi_X(u) &= \int_0^1 (\phi_X(uw))^2 dw \\ &= \frac{1}{u}\int_0^1 \phi^2_X(w)dw. \end{aligned}$$

where $B(.,.)$ is the Beta-function. Suppose $X_1 \sim \Gamma(1,a)$ and $X_2 \sim \Gamma(1,a)$ and X_1 and X_2 are independent. Then the Laplace transform of $X_1 + X_2$ is

$$\phi_{X_1+X_2}(s) = \left(\frac{1}{1+s}\right)^{2a}$$

and the Mellin transform of this Laplace transform is

$$B(2a-s,s) = \frac{\Gamma(s)\Gamma(2a-s)}{\Gamma(2a)}.$$

From the result obtained in Section 2, we observe that the Mellin transform of W is

$$\begin{aligned} \hat{f_W}(s) &= \frac{\hat{\phi}_g(1-s)}{\hat{\phi}_g^2(1-s)} \\ &= \frac{B(a-(1-s),(1-s))}{B(2a-(1-s),(1-s))} \\ &= \frac{\Gamma(1-s)\Gamma(a+s-1)}{\Gamma(a)} \frac{\Gamma(2a)}{\Gamma(2a+s-1)\Gamma(1-s)} \\ &= \frac{\Gamma(2a)\Gamma(a+s-1)}{\Gamma(s)\Gamma(2a+s-1)}. \end{aligned}$$

Using the inversion formula for Mellin transforms, we obtain that

$$f_W(w) = \frac{1}{B(a,a)} w^{a-1}(1-w)^{a-1}, \quad 0 \le w \le 1$$

which in turn proves that the random variable $W \sim \text{Beta}(a,a)$, that is, W has the Beta distribution with parameters a and a.

Remarks 4.2:(i) If, in the above theorem, we assume that (i) X, X_1 and X_2 are independent but not necessarily identically distributed such that $X \sim \Gamma(1,a)$, $X_1 \sim \Gamma(1,a)$ and $X_2 \sim \Gamma(1,b)$, then , it can be shown that $W \sim \text{Beta}(a,b)$. This can be seen from the following relations. Note that

$$\begin{aligned} \hat{\phi_{X_1}}(s) &= \frac{\Gamma(s)\Gamma(a-s)}{\Gamma(a)}, \\ \hat{\phi_{X_1+X_2}}(s) &= \frac{\Gamma(s)\Gamma(a+b-s)}{\Gamma(a+b)}, \end{aligned}$$

and hence

$$\begin{aligned} \hat{f_W}(s) = \frac{\phi\hat{X}_1(1-s)}{\hat{\phi_{X_1+X_2}}(1-s)} &= \frac{\Gamma(1-s)\Gamma(a+s-1)}{\Gamma(a)} \frac{\Gamma(a+b)}{\Gamma(1-s)\Gamma(a+b+s-1)} \\ &= \frac{\Gamma(a+b)\Gamma(a+s-1)}{\Gamma(a)\Gamma(a+b+s-1)}. \end{aligned}$$

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