METRON - *International Journal of Statistics*
2005, vol. LXIII, n. 3, pp. 529-559

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Inverse stochastic orders and generalized Gini functionals

Summary - We investigate the class of stochastic orders induced by Generalized Gini Functionals (GGF) [Yaari (1987) dual functionals] and identify the maximal classes of functionals associated with these orders. Our results are inspired by Marshall (1991) and are dual to those obtained for additive representations in Müller (1997) and in Castagnoli and Maccheroni (1998). The closure of the convex hull generated by a given set of probability distortion functions ($\mathcal{F}$) [or by a set of rank-dependent weighting functions ($\mathcal{V}$)] identifies the maximal class of functionals associated with the stochastic orders that are consistent with $\mathcal{F}$ [or $\mathcal{V}$]. Rank-dependent weighting functions obtained as convex combinations of indicator functions identify GGFs that can be considered the basis of relevant stochastic orders in decision theory and inequality measurement. As hinted by Wang and Young (1998) and Zoli (1999, 2002) the stochastic orders obtained are related to the class of inverse stochastic dominance (ISD) conditions introduced in Muliere and Scarsini (1989). Making use of our results we review some stochastic dominance conditions that can be applied in decision theory, inequality, welfare and poverty measurement. These conditions are associated with orders implied by first order ISD and implying second order ISD, as well as with orders implied by the latter.

Key Words - Stochastic orders; Inverse stochastic dominance; Lorenz dominance; Generalized Gini indices; Inequality measurement; Welfare measurement.

1. Introduction

The seminal work of Gini (1914) on the measurement of concentration of income distributions has stimulated a large literature on inequality, welfare and poverty measurement and in decision theory [see Sen, 1973 and Lambert, 2001]. The close and direct relationship between the *Gini index* and the *Lorenz curve* has been one of the main reasons for the wide application of the index in empirical analysis. However, from the theoretical point of view the index has been criticized because of its inconsistence with the orders induced by utilitarian (i.e. additively separable) evaluation functions (see Newbery, 1970).

Received September 2005.

As argued in Yaari (1987, 1988) and Weymark (1981) the appropriate normative framework of analysis that is consistent with the Gini index is the *dual approach* where income distributions are evaluated according to weighted averages of incomes ranked in increasing order and weighted according to their positions. In decision theory the formulation of Yaari *dual functionals* (also known as Generalized Gini functionals) is more commonly known as averages of probability distributions where probabilities are distorted. As shown in Muliere and Scarsini (1989), Wang and Young (1998), Zoli (1999), Chateauneuf *et al.* (2000, 2002) and Aaberge (2004) the stochastic orders induced by Yaari *functionals* are related with the concept of *inverse stochastic dominance* introduced in Muliere and Scarsini (1989).

In this work we focus on Yaari *functionals* and consider the stochastic orders obtained as unanimous dominance of these functionals for a given set of admissible probability distortions, or alternatively considering sets of weighting functions. For any set of distortion [weighting] functions we are interested in identifying the *maximal set* of distortions [weights] inducing the same stochastic order. Our results are inspired by Marshall (1991) and are dual to those obtained for additive representations in Müller (1997) and in Castagnoli and Maccheroni (1998). Given a set of policy makers with contrasting views, formalized by differing probability distortions, any policy maker exhibiting a distortion function belonging to the *closure of the convex hull* of those of the formers will evaluate distributions consistently with them. A related result is derived also for weighting functions. Our methodology will also make possible to investigate the normative relevance of small sets of distortion [weighting] functions that can constitute the *bases* for a given stochastic order. In particular, for weighting functions (more common in the literature on inequality and welfare measurement) it is possible to identify bases formed by families of transformations or combinations of indicator functions specified in the quantiles space. These bases are shown to induce stochastic orders that are equivalent to the inverse stochastic dominance conditions or to some other alternative conditions specified making use of thresholds partitioning income distributions into groups identified by the positions in the income ranking.

1.1. Preliminaries and notation

We consider stochastic dominance conditions and related stochastic orders associated with Yaari (1987) functionals.

Let $(\Omega, \mathcal{A}, P)$ be a probability space. A random variable (income distribution) is a measurable function $X : \Omega \to \mathbb{R}_+$. We assume that P is adequate (i.e. either P is non-atomic or Ω is finite and P is uniform).

Considering the analogy between income distributions and random variables, Ω could be interpreted as a population (either discrete or continuous) of individuals and X as a *non-negative bounded* random variable. In our framework $X(\omega)$ denotes the income level of agent $\omega \in \Omega$, therefore $X \in \mathcal{L}_\infty^+$ is called *income profile.* If $\Omega := \{\omega_1, \omega_2, \dots \omega_i, \dots \omega_n\}$, and a uniform P attaches equal mass $1/n$ to each ω_i, then we have the *n–dimensional empirical case* represented by the vector $\mathbf{x} := (x_1, x_2, \dots x_i, \dots x_n)$ where $x_i := X(\omega_i)$. Let $F_X(t)$ denote the cumulative distribution function of an income profile X with bounded support and finite mean $\mu(X) = \int_0^{+\infty} t \, dF_X(t)$. The *decumulative distribution function* is $\bar{F}_X(t) := P\{X > t\} = 1 - F_X(t)$, it is always non-increasing and right-continuous. Moreover, we let

$$F_X^{-1}(p) := \inf\{x \in \mathbb{R}_+ : F_X(x) \geq p\} \quad \text{with} \quad 0 \leq p \leq 1$$

be the *left continuous inverse distribution* function showing the income of an individual at the p quantile of the distribution. Denote by $C([0,1])$ the set of all continuous functions on the unit interval $[0,1]$, endowed with the supnorm. A non-decreasing function f in $C([0,1])$ such that $f(0) = 0$ and $f(1) = 1$, is called *probability distortion.*

Given two random variables X and Y, and a family $\mathcal{F}$ of probability distortions f, we consider the class of *stochastic orders* of the form

$$X \geqslant_{\mathcal{F}} Y \Leftrightarrow \int_\Omega X d(f \circ P) \geq \int_\Omega Y d(f \circ P) \quad \forall f \in \mathcal{F} \tag{1}$$

introduced in Yaari (1987, 1988) and axiomatically characterized in Maccheroni (2004). Dually to what done in Müller (1997) and Castagnoli and Maccheroni (1998) for integral stochastic orders, in this note we characterize *the maximal set* of distortions representing a given stochastic order. This problem was first raised for integral stochastic orders in Marshall (1991).

As observed by Chateauneuf, Cohen, and Meilijson (1997) and Chateauneuf and Moyes (2004, 2005) several orders used to rank the riskiness of random variables or the dispersion of income distributions exhibit this form. For example, Rothschild and Stiglitz (1970)'s *second order stochastic dominance* is obtained when $\mathcal{F}$ is the set of all convex probability distortions; Bickel and Lehman (1976)'s *dispersion* is obtained when $\mathcal{F}$ is the set of all probability distortions that are majorized by the identity; Jewitt (1989)'s *location independent riskiness* is obtained when $\mathcal{F}$ is the set of all probability distortions that are star-shaped at 1, and Muliere and Scarsini (1989) *third order inverse stochastic dominance* is obtained when $\mathcal{F}$ is the set of all probability distortions with non-decreasing second derivative [see Zoli, 1999 and Chateauneuf, Gajdos and Wilthien, 2002].

In this work we focus on comparisons of income distributions and analyze (i) stochastic *dominance* conditions $\succcurlyeq$ obtained comparing directly transformations of income distributions and (ii) stochastic *orders* $\geqslant$ based on (1). In particular we express the stochastic order $\geqslant_{\mathcal{F}}$ in terms of the class of *Generalized Gini Social Evaluation Functions (GG-SEFs)* also known as generalized Gini functionals which is commonly used in the theoretical and applied literature on income distributions. The *GG-SEFs* have been introduced in Weymark (1981) and consider weighted averages of incomes ranked in increasing order and *weighted* according to their positions [see also Ebert, 1988, Ben Porath and Gilboa, 1994 and Safra and Segal, 1998].

To be more precise, we restrict attention to *absolutely continuous* distortion functions f. We let $V(1-p) = 1 - f(p)$ for $p \in [0, 1]$ and consider the set of weights $v(p) \geq 0$ for $p \in [0, 1]$ such that $V(p) = \int_0^p v(t)dt$. Thus the *Generalized Gini SEFs* can be expressed as

$$W_v(X) := \int_0^1 v(p) \cdot F_X^{-1}(p)dp, \tag{2}$$

where $v(p) \geq 0$ denotes the weight attached to the p quantile.(1) A well known subset of the *GG-SEFs* is the class of S-Gini [single parameter] SEFs $\Xi(\delta; .)$ introduced in Donaldson and Weymark (1980, 1983) and Yitzhaki (1983). This class of SEFs is parameterized by $\delta \geq 1$ and is obtained letting $v(p) = \delta(1-p)^{\delta-1}$ that is

$$\Xi(\delta; X) := \delta \int_0^1 (1-p)^{\delta-1} F_X^{-1}(p)dp. \tag{3}$$

Note that $\Xi(1; X) = \mu(X)$ while for $\delta = 2$ we obtain the SEF associated with the Gini index $G(.)$ i.e. $\Xi(2; X) = \mu(X) \cdot [1 - G(X)]$. In general we can derive the (relative) S-Gini index of inequality $G(\delta; X)$ parameterized by δ as

$$G(\delta; X) := 1 - \frac{\Xi(\delta; X)}{\mu(X)} = \int_0^1 \left[1 - \delta(1-p)^{\delta-1}\right] \frac{F_X^{-1}(p)}{\mu(X)} dp.$$

Throughout the paper we suppose that the weights are *non-negative* and *integrate to 1*, i.e. $v \in \mathcal{V}^0$ where

$$\mathcal{V}^0 := \{v \in \mathcal{L}_1([0, 1]) : v \geq 0, \text{ and } \int_0^1 v(t)dt = 1\}.$$

In general, a class $\mathcal{V} \subseteq \mathcal{V}^0$ will represent an *admissible set of weighting functions*, thus under the assumption of absolutely continuous f's then (1) can be equivalently restated in terms of (2) as:

$$X \geqslant_{\mathcal{V}} Y \Leftrightarrow W_v(X) \geq W_v(Y) \quad \forall v \in \mathcal{V}. \tag{4}$$

(1) For the empirical case the *GG-SEFs* reduces to $W_V^n(\mathrm{x}) = \sum_{i=1}^n \left[V\left(\frac{i}{n}\right) - V\left(\frac{i-1}{n}\right)\right] \cdot x_{(i)}$ where incomes $x_{(i)}$ are welfare ranked income realizations such that $x_{(i)} \leq x_{(i+1)}$.

Our aim is to investigate the relation between the stochastic order $\geqslant_\nu$ and stochastic dominance conditions.

The most common *stochastic dominance* condition in inequality measurement is the partial ranking criterion based on *Lorenz dominance*. An income profile shows no less dispersion than another, in terms of income shares, if its Lorenz curve is nowhere below that of the other profile.(2) Making use of Gastwirth (1971), the Lorenz curve for X is defined as:

$$L_X(p) = \int_0^p \frac{F_X^{-1}(t)}{\mu(X)} dt.$$

We will say that income profile X *Lorenz dominates* income profile Y, $X \succcurlyeq_L Y$, if and only if

$$L_X(p) \geq L_Y(p) \text{ for all } p \in [0, 1] .$$

If $\mu(X) = \mu(Y)$ this condition is equivalent to *second degree stochastic dominance*, which in turn is equivalent to *generalized Lorenz dominance* obtained comparing generalized Lorenz curves $GL_X(p) := \mu(X) \cdot L_X(p)$. Generalized Lorenz dominance of X w.r.t. Y requires to check that the following condition is satisfied (Kolm, 1969 and Shorrocks, 1983):

$$\int_0^p \Delta(t)dt \geq 0 \text{ for all } p \in [0, 1]$$

where $\Delta(p) = \Delta^1(p) := F_X^{-1}(p) - F_Y^{-1}(p)$.

The previous dominance conditions are obtained making use of comparisons of integrals of inverse distribution functions, they are special cases of the family of *Inverse Stochastic Dominance* (ISD) conditions introduced in Muliere and Scarsini (1989). Let $\Delta^i(p) := \int_0^p \Delta^{i-1}(t)dt$ for $i = 2, 3, \ldots$ The *ISD* condition of order i ($\succcurlyeq_i$) is obtained comparing the integral of the inverse distribution functions derived recursively.

Definition 1.1. (Inverse Stochastic Dominance) $X \succcurlyeq_i Y$ iff $\Delta^i(p) \geq 0$ for all $p \in [0, 1]$.

Note that $X \succcurlyeq_1 Y$ denotes *rank dominance* (Saposnik, 1981) while $X \succcurlyeq_2 Y$ denotes *generalized Lorenz dominance*.(3)

In what follows, after deriving the main results on the maximal set and the bases of stochastic orders, we will review and investigate welfare comparisons between distributions with different total means. In particular we identify bases

(2) See Atkinson (1970), Kolm (1969), Sen (1973), Dasgupta, Sen and Starrett (1973), Rothschild and Stiglitz (1973) and Fields and Fei (1978).

(3) As argued in Muliere and Scarsini (1991) standard direct stochastic dominance and ISD are equivalent only for first and second order.

for ISD of order n and show that they are related with dominance for the class of S-Ginis in (3) for a value of the parameter $\delta = n - 1$ when applied to distributions truncated at quantiles p.

Then we will move to analyse stochastic orders based on specifications of weighting functions that take into account population thresholds. It might be of interest to point out that some dominance conditions obtained in next sections will make use of comparisons of *incomplete means* that is of the functions $GL(p)/p$ measuring the average income of the subgroup of the p poorest individuals. The value of $\Delta GL(p)/p$ can also be interpreted as the ratio of the difference between the *average income gap* of the two distributions truncated at quantile p when poverty line is fixed at a value $z \geq \max\{F_X^{-1}(p), F_Y^{-1}(p)\}$ and the headcount ratio of both distributions that coincides with p.

2. Maximal generators for dual stochastic orders

For a given stochastic order $\geqslant_{\mathcal{F}}$ as specified in (1) our main concern is to identify the *maximal set* of distortion functions that is consistent with $\geqslant_{\mathcal{F}}$. That is, we aim at obtaining the larger possible set of views on the ranking of distributions, formalized by the specification of distortion functions, that is consistent with the stochastic order, and therefore is consistent with the views specified by all distortion functions in $\mathcal{F}$.

An implication of the result will allow to highlight conditions under which it is possible to identify *small sets* of distortion functions inducing a given stochastic order. We call these sets *bases* for the stochastic order. The selection of a *base* inducing $\geqslant_{\mathcal{F}}$ will allow to make explicit the most elementary ethical views underlying the stochastic order.

Building upon the results obtained for stochastic orders $\geqslant_{\mathcal{F}}$ in (1) we also present analogous results that are valid for the stochastic orders $\geqslant_{\mathcal{V}}$ in (4).

2.1. Main theorems

Let $\mathcal{L}_{\infty}^{+}$ denote the set of all *non-negative bounded* random variables. We say that a family $\mathcal{G}$ of probability distortions *represents* the stochastic order $\geqslant_{\mathcal{F}}$ defined in (1) if and only if $\geqslant_{\mathcal{G}}$ coincides with $\geqslant_{\mathcal{F}}$; moreover we say that $\geqslant_{\mathcal{G}}$ *is weaker than* $\geqslant_{\mathcal{F}}$ if and only if, for all $X, Y \in \mathcal{L}_{\infty}^{+}$,

$$X \geqslant_{\mathcal{F}} Y \Rightarrow X \geqslant_{\mathcal{G}} Y.$$

We denote by $\overline{co}(\mathcal{F})$ the *(supnorm) closed convex hull* of a subset $\mathcal{F}$ of $C([0, 1])$. We are now ready to state our first results proved in the Appendix.

Theorem 2.1. *For every set $\mathcal{F}$ of probability distortions, the maximal family of probability distortions representing $\geqslant_{\mathcal{F}}$ is $\overline{co}(\mathcal{F})$.*

Corollary 2.1. *Given two sets $\mathcal{F}$ and $\mathcal{G}$ of probability distortions, $\geqslant_{\mathcal{G}}$ is weaker than $\geqslant_{\mathcal{F}}$ if and only if $\overline{co}(\mathcal{G}) \subseteq \overline{co}(\mathcal{F})$. In particular, $\geqslant_{\mathcal{F}}$ and $\geqslant_{\mathcal{G}}$ coincide if and only if $\overline{co}(\mathcal{G}) = \overline{co}(\mathcal{F})$.*

The obtained results are valid for all distributions in $\mathcal{L}_{\infty}^{+}$ and all sets of distortion functions in $C([0,1])$. Theorem 2.1 formalizes the intuition that given two different distortion functions any distortion function obtained as a convex combination of those ranks distributions consistently with the stochastic order induced by them. Indeed the theorem generalizes the intuition to all stochastic orders and associated sets of distortion functions focussing on the closure of their convex hull. Moreover it makes evident that *only* the distortion functions in $\overline{co}(\mathcal{F})$ can be consistent with $\geqslant_{\mathcal{F}}$.

From Theorem 2.1 it follows that any stochastic order is represented by a *maximal set* of distortions, and this maximal set is unique. Thus if two stochastic orders exhibit the same maximal set then they coincide. Therefore all subsets of an initial set of distortion functions generating the same closed and convex set can be considered as a *base* for the associated stochastic order.

Theorem 2.1 and Corollary 2.1 provide neat results for the stochastic orders in (1) expressed in terms of distortion functions, however, our main concern is to make use of these results to derive analogous results for the associated class of stochastic orders in (4).

For a given stochastic order, focussing on bases will allow to make explicit the underlying normative assumptions and to easily implement the order. The specification in (4) makes use of the weighting functions to formalize different concerns on incomes based on their rank. The possibility to select bases of weighting functions (instead of distortion functions) inducing a given stochastic order will therefore provide a useful tool to evaluate the normative relevance of the associated stochastic order. Expressing the normative content of a stochastic order in terms of small sets of weights seems in our view more appealing for the evaluation of income distributions since it makes explicit use of two important informations: the *relative weight* associated with an income within the distribution and *its relative position*.

The main concern related to the application of the results in Theorem 2.1 to the stochastic orders in (4) is that while distortions functions belong in general to the class $C([0,1])$, those associated with weighting functions are absolutely continuous functions on $[0,1]$.

Let $AC([0,1])$ denote the set of *absolutely continuous* functions on $[0,1]$. If f is absolutely continuous, standard analysis results guarantee that:

(i) f is almost everywhere differentiable on $[0,1]$, $f' \in \mathcal{L}_{1}^{+}([0,1])$, and $f(p) = \int_0^p f'(t)\,dt$ for all $p \in [0,1]$.

(ii) For all $X \in \mathcal{L}_{\infty}^{+}$, $\int_{\Omega} X d(f \circ P) = \int_0^1 f'(1-t) \cdot F_X^{-1}(t)\,dt$.

In this case, $v_f(t) = f'(1-t)$ is called *weighting function associated with* f. In general, a *weighting function* v is an element of $\mathcal{L}_1^+([0,1])$ such that $\int_0^1 v(t)\,dt = 1$; the *associated probability distortion* is

$$f_v(p) = \int_0^p v(1-t)\,dt = \int_{1-p}^1 v(s)\,ds, \qquad \forall p \in [0,1].$$

Instead of a set $\mathcal{F}$ of probability distortions we take as primitive concept the set $\mathcal{V}$ of weighting functions and put

$$\mathcal{F} = \mathcal{F}_{\mathcal{V}} := \{f_v : v \in \mathcal{V}\}.$$

In this case we write $\geqslant_{\mathcal{V}}$ instead of $\geqslant_{\mathcal{F}_{\mathcal{V}}}$. For every set $\mathcal{V}$ of weighting functions, we put

$$\widehat{co}(\mathcal{V}) := \{f'(1-t) : f \in \overline{co}(\{f_v : v \in \mathcal{V}\}) \cap AC([0,1])\},$$

that is we identify all absolutely continuous distortion functions belonging to the closure of the convex hull of the distortion functions derived starting from $\mathcal{V}$, and we consider the set of all weighting functions that are associated with them.

Proposition 2.1. *Let $\mathcal{V}$, $\mathcal{W}$ be sets of weighting functions, $\geqslant_{\mathcal{W}}$ is weaker than $\geqslant_{\mathcal{V}}$ if and only if $\mathcal{W} \subseteq \widehat{co}(\mathcal{V})$. In particular,*

(1) *$\widehat{co}(\mathcal{V})$ is the maximal set of weighting functions that represents $\geqslant_{\mathcal{V}}$;*
(2) *$\geqslant_{\mathcal{V}}$ and $\geqslant_{\mathcal{W}}$ coincide if and only if $\widehat{co}(\mathcal{W}) = \widehat{co}(\mathcal{V})$.*

The idea behind the proposition is very simple: if $\geqslant_{\mathcal{F}_{\mathcal{W}}}$ is weaker than $\geqslant_{\mathcal{F}_{\mathcal{V}}}$ then $\mathcal{F}_{\mathcal{W}} \subseteq \overline{co}(\mathcal{F}_{\mathcal{W}}) \subseteq \overline{co}(\mathcal{F}_{\mathcal{V}})$ and from

$$\overline{co}(\mathcal{F}_{\mathcal{W}}) \cap AC([0,1]) \subseteq \overline{co}(\mathcal{F}_{\mathcal{V}}) \cap AC([0,1])$$

it follows that $\mathcal{W} \subseteq \widehat{co}(\mathcal{W}) \subseteq \widehat{co}(\mathcal{V})$. Conversely, if $\mathcal{W} \subseteq \widehat{co}(\mathcal{V})$, then

$$\mathcal{F}_{\mathcal{W}} \subseteq \mathcal{F}_{\widehat{co}(\mathcal{V})} = \overline{co}(\{f_v : v \in \mathcal{V}\}) \cap AC([0,1]) \subseteq \overline{co}(\mathcal{F}_{\mathcal{V}}),$$

and hence $\overline{co}(\mathcal{F}_{\mathcal{W}}) \subseteq \overline{co}(\mathcal{F}_{\mathcal{V}})$, which in turn implies that $\geqslant_{\mathcal{W}}$ is weaker than $\geqslant_{\mathcal{V}}$. See the Appendix for details.

The practical verification that $\geqslant_{\mathcal{W}}$ is weaker than $\geqslant_{\mathcal{V}}$ then goes as follows: if $\mathcal{W} \subseteq co(\mathcal{V})$, we are done; else we have to check that for all $w \in \mathcal{W}$ there exists a sequence $v^n \in co(\mathcal{V})$ such that $f_{v^n} \to f_w$ uniformly; that is $\int_{1-p}^1 v^n(t)\,dt \to \int_{1-p}^1 w(t)\,dt$ uniformly w.r.t. $p \in [0,1]$. For example this is guaranteed by $v^n \to w$ in $\mathcal{L}_1^+([0,1])$. Then:

Corollary 2.2. *For every $\mathcal{V} \subseteq \mathcal{V}^0$ we have $\overline{co}^{\mathcal{L}_1}(\mathcal{V}) \subseteq \widehat{co}(\mathcal{V})$. In particular if $\mathcal{B} \subseteq \mathcal{V}$, and for every $v \in \mathcal{V}$ there exists a sequence v_n in $co(\mathcal{B})$ that converges to v in $\mathcal{L}_1([0,1])$, then $\geqslant_{\mathcal{B}}$ coincides with $\geqslant_{\mathcal{V}}$.*

Therefore according to Proposition 2.1 a proper candidate for a base for $\geqslant_{\mathcal{V}}$ is any $\mathcal{W}$ such that $\widehat{co}(\mathcal{W}) = \widehat{co}(\mathcal{V})$. However, Corollary 2.2 shows that in practice, taking the closure of $co(\mathcal{V})$ in $\mathcal{L}_1([0,1])$, could be more convenient and allows to identify more easily verifiable bases. This is precisely the approach we follow in next section.

3. Orderings of income distributions

In order to investigate the relationships between stochastic orders $\geqslant_{\mathcal{V}}$ based on GG-SEFs and inverse stochastic dominance conditions it will be first useful to recall that according to (2) and (4)

$$X \geqslant_{\mathcal{V}} Y \Longleftrightarrow \int_0^1 v(p) \cdot \Delta(p)dp \geq 0 \ \forall v \in \mathcal{V}.$$

Following the results in Theorem 2.1, Corollary 2.1 and Proposition 2.1 we know that if $X \geqslant_{\mathcal{V}} Y$ then the weighting function $\hat{v}$ *preserves* $\geqslant_{\mathcal{V}}$ [i.e. $X \geqslant_{\mathcal{V}} Y \Longrightarrow X \geqslant_{\hat{v}} Y$] if and only if $\hat{v} \in \widehat{co}(\mathcal{V})$. Combining this result with the fact that $\overline{co}(\mathcal{V}) \subseteq \widehat{co}(\mathcal{V})$ for any $\mathcal{V}$,(4) as in Corollary 2.2, it is then possible to make explicit the following remarks that will allow to work directly with the closures of the convex hull of sets of weights instead of moving from weights to distortions and coming back as required in Proposition 2.1.

Remark 3.1. $\hat{v}$ preserves $\geqslant_{\mathcal{V}}$ if $\hat{v} \in \overline{co}(\mathcal{V})$.

Remark 3.2. In order to derive $X \geqslant_{\mathcal{V}} Y$ it is sufficient to identify a set of weights $\mathcal{B}$ s.t. $\mathcal{B} \subseteq \overline{co}(\mathcal{V})$ and $\mathcal{V} \subseteq \overline{co}(\mathcal{B})$. According to Remark 3.1 condition $X \geqslant_{\mathcal{B}} Y$ will be sufficient to guarantee $X \geqslant_{\mathcal{V}} Y$ since $\mathcal{V} \subseteq \overline{co}(\mathcal{B})$ moreover, it will be also necessary since $\mathcal{B} \subseteq \overline{co}(\mathcal{V})$.

Therefore, considering stochastic orders induced by subsets $\mathcal{V}$ of the set $\mathcal{V}^0$ of normalized non-negative weights will require to restrict attention to the closure of the convex hull generated by the weights in $\mathcal{V}$.

In order to investigate the relationships between orders $\geqslant_{\mathcal{V}}$ where $\mathcal{V} \subseteq \mathcal{V}^0$ we will make direct use of classes of weights $\mathcal{B}$ that can be considered bases for $\mathcal{V}$.

Definition 3.1. (Bases for $\geqslant_{\mathcal{V}}$) Let $\mathcal{V} \subseteq \mathcal{V}^0$, a set of weights $\mathcal{B} \subseteq \mathcal{V}^0$ is a base for $\geqslant_{\mathcal{V}}$ if $\mathcal{B} \subseteq \overline{co}(\mathcal{V})$ and $\mathcal{V} \subseteq \overline{co}(\mathcal{B})$.

(4) Since $\mathcal{V} \subset \mathcal{L}_1([0,1])$ we write $\overline{co}(\mathcal{V})$ to denote the convex closure of $\mathcal{V}$ in $\mathcal{L}_1([0,1])$.

Thus according to Remark 3.2 we have that $\geqslant_{\mathcal{B}} \longleftrightarrow \geqslant_{\mathcal{V}}$.

For a given stochastic order $\geqslant_{\mathcal{V}}$ it is particularly interesting to focus on bases such that $\mathcal{B} \subset \mathcal{V}$ trying to identify *minimal sets* of weighting functions inducing $\geqslant_{\mathcal{V}}$. Unfortunately while the maximal set of $\geqslant_{\mathcal{V}}$ is unique, this is *not* the case for the bases, moreover in general bases can be identified by non overlapping sets of weights therefore it might *not be possible to identify a minimal set* associated with a stochastic order as in next example.

Example 1. Let $\mathcal{B}_D^0$ denote the set of all discontinuous functions in $\mathcal{V}^0$ and $\mathcal{B}_C^0$ denote the set of all continuous functions in $\mathcal{V}^0$ i.e. $\mathcal{B}_C^0 := \mathcal{V}^0 \backslash \mathcal{B}_D^0$. Then $\overline{co}(\mathcal{B}_D^0) = \overline{co}(\mathcal{B}_C^0) = \overline{co}(\mathcal{V}^0)$ thus $\geqslant_{\mathcal{B}_D^0} \Leftrightarrow \geqslant_{\mathcal{B}_C^0} \Leftrightarrow \geqslant_{\mathcal{V}^0}$.

Given the general impossibility to identify bases that are minimal sets for a stochastic order we will focus mainly on specific subsets of $\mathcal{V}^0$ that will allow to provide a more intuitive interpretation of the results. We will construct most bases making use of the functions $I_{[0,\alpha)}(p)$ denoting the indicator function on $[0, \alpha)$ i.e.

$$I_{[0,\alpha)}(p) := \begin{cases} 1 & \text{for } p \in [0, \alpha) \\ 0 & \text{for } p \in [\alpha, 1]. \end{cases}$$

Interpretation of the set of weights obtained combining transformations of indicator functions is, in our opinion, often immediate since it involves directly (i) the *identification* of a range of income units in the income ranking space and (ii) the *selection of a weight* associated with the incomes belonging to these units.

Next results identify bases for the ISD orders and will motivate also comparisons based on transformations of the $\Delta^i(p)$ functions. In particular the differences of incomplete means derived as $\Delta^2(p)/p$ will play a crucial role in the analysis.

3.1. *Bases and generators of Inverse Stochastic Dominance*

In this subsection we highlight the application of the procedure derived in the previous section to the derivation of the ISD conditions.

We consider the set $\mathcal{V}^1 \subset \mathcal{V}^0$ obtained selecting from $\mathcal{V}^0$ only weighting functions such that $v(1) = 0$,(5) that is

$$\mathcal{V}^1 := \{v \in \mathcal{V}^0 : v(1) = 0\}.$$

The bases for $\geqslant_{\mathcal{V}^1}$ are given by the set $\mathcal{B}^1 \subset \mathcal{V}^1$ of all functions assigning constant positive weight to any interval of values $[\ell, h)$, that is all

$$b_1(p, \ell, h) := 1/(h - \ell) \cdot \left[I_{[0,h)}(p) - I_{[0,\ell)}(p)\right] \text{ where } 1 > h > \ell \geq 0. \quad (5)$$

(5) Note that by definition $\mathcal{V}^0 = \mathcal{V}^1$in $\mathcal{L}_1^+([0, 1])$. Making explicit condition $v(1) = 0$ will turn out to be useful for the recursive derivation of the set $\mathcal{V}^n$.

As can be intuitively conjectured (see also formal derivation in Lemma A.1 in the Appendix) applying $\geqslant_{\mathcal{B}^1}$ we obtain $\geqslant_{\mathcal{V}^1}$ and easily it can be checked that we obtain also $\succcurlyeq_1$. Recalling known results by Mehran (1976) and Yaari (1987, 1988) where $\succcurlyeq_1 \longleftrightarrow \geqslant_{\mathcal{V}^1}$ we can summarize these findings as:

Proposition 3.1. $X \geqslant_{\mathcal{V}^1} Y \Longleftrightarrow X \geqslant_{\mathcal{B}^1} Y \Longleftrightarrow X \succcurlyeq_1 Y.$

In order to restrict attention to inequality averse evaluations we consider the set $\mathcal{V}^2 \subset \mathcal{V}^1$ of non increasing weights [see Mehran, 1976, Yaari, 1987, 1988, Chateauneuf *et al.* 2002]. That is

$$\mathcal{V}^2 := \{v \in \mathcal{V}^1 : v \text{ non-increasing}\}.$$

As shown in the Appendix (see Lemma A.2) the bases for $\geqslant_{\mathcal{V}^2}$ are given by the set $\mathcal{B}^2$ of functions assigning constant positive (and normalized) value to an interval of positions $[0, h)$, that is

$$b_2(p, h) := 1/h \cdot I_{[0,h)}(p) \text{ where } h \in (0, 1). \tag{6}$$

Applying $\geqslant_{\mathcal{B}^2}$ we obtain $\succcurlyeq_2$ therefore we have [see Mehran, 1976, Yaari, 1987, 1988, Chateauneuf *et al.* 2002]:

Proposition 3.2. $X \geqslant_{\mathcal{V}^2} Y \Longleftrightarrow X \geqslant_{\mathcal{B}^2} Y \Longleftrightarrow X \succcurlyeq_2 Y.$

We now show a trivial result based on a restatement of the definition of the generalized Lorenz curve that will open the way to the interpretation of ISD in terms of GG-SEFs dominance for distributions truncated at quantile p. Let X_p denote the income profile X truncated at quantile p such that $F_{X_p}^{-1}(t) = F_X^{-1}(t \cdot p)$, then $\mu(X_p)$ is the incomplete mean of distribution X evaluated for the poorest p proportion of individuals. Direct application of $\geqslant_{\mathcal{B}^2}$ leads to

$$\int_0^1 1/h \cdot I_{[0,h)}(t) \cdot \Delta(t)dt = 1/h \int_0^1 \Delta_h(t)dt \geq 0 \tag{7}$$

for all $h \in (0, 1)$ where $\Delta_h(t) := F_{X_h}^{-1}(t) - F_{Y_h}^{-1}(t) = \Delta(t \cdot h)$.

The following trivial result expresses $\succcurlyeq_2$ in terms of comparisons of incomplete means of distributions X and Y evaluated for all proportions p of poorest individuals.

Remark 3.3. $\Xi(1; X_p) = \mu(X_p) \geq \Xi(1; Y_p) = \mu(Y_p)$ for all $p \in [0, 1] \Longleftrightarrow X \succcurlyeq_2 Y.$

Next set of weights allows to characterize third degree ISD. We consider the set $\mathcal{V}^3 \subset \mathcal{V}^2$ of convex weights [see Mehran, 1976, Kakwani, 1980, Zoli, 1999, and Chateauneuf *et al.* 2002] exhibiting zero left-hand side derivative v'_- at the top. That is

$$\mathcal{V}^3 := \{v \in \mathcal{V}^2 : v \text{ convex}, \; v'_-(1) = 0\}.$$

The bases for $\geqslant_{\mathcal{V}^3}$ are given by the set $\mathcal{B}^3$ of all linear decreasing weighting functions censored for values below 0, that is

$$b_3(p,h) := \frac{2}{h^2}\int_p^1 I_{[0,h)}(t)dt = \frac{2(h-p)_+}{h^2} \text{ where } h \in (0,1), \tag{8}$$

and $(h-p)_+ := \max\{h-p, 0\}$ (see Lemma A.3 in Appendix). Making use of $\geqslant_{\mathcal{B}^3}$ we obtain $\geqslant_{\mathcal{V}^3}$ and therefore adapting results by Zoli (1999) and Wang and Young (1998) showing that $\geqslant_{\mathcal{V}^3} \leftrightarrow \succcurlyeq_3$ we can prove:

Proposition 3.3. $X \geqslant_{\mathcal{V}^3} Y \iff X \geqslant_{\mathcal{B}^3} Y \iff X \succcurlyeq_3 Y$.

Straightforward readjustments of variables lead to a restatement of $\succcurlyeq_3$ in terms of the stochastic order induced by the single parameter GG-SEFs $\Xi(2; X)$ that is by those related to the Gini index (see also Zoli 1999, 2002). Application of $\geqslant_{\mathcal{B}^3}$ requires that

$$\int_0^1 \frac{2(h-t)_+}{h^2} \cdot \Delta(t)dt = \int_0^1 2(1-t) \cdot \Delta_h(t)dt \geq 0 \tag{9}$$

for all $h \in (0,1)$. That is recalling that the second part of (9) coincides with $\Xi(2; X_h) - \Xi(2; Y_h) = \mu(X_h)[1 - G(X_h)] - \mu(Y_h)[1 - G(Y_h)]$ we get as in Zoli (1999, 2002).

Remark 3.4. $\Xi(2; X_p) \geq \Xi(2; Y_p)$ for all $p \in [0,1] \iff X \succcurlyeq_3 Y$.

The previous result points out that $\geqslant_{\mathcal{B}^3}$ coincides with dominance for all Gini based SEFs applied to any set of poorest p proportions of individuals.

For higher orders (discussed in Wang and Young, 1998 and Aaberge, 2004), in analogy with what suggested for direct stochastic dominance [see Fishburn and Willig, 1984], when $n > 2$ we have to consider the set

$$\mathcal{V}^{*n} := \{v \in \mathcal{V}^{*n-1} : -v'_- \in \mathcal{V}^{*n-1}\}$$

obtained recursively starting from $\mathcal{V}^{*2} := \{v : v \geq 0, \; v \text{ is non-increasing}, \; v(1) = 0\}$ and derive the set of interest

$$\mathcal{V}^n := \{v \in \mathcal{V}^{*n} : \int_0^1 v(t)dt = 1\}$$

which takes into account normalized weights . In general the bases $\mathcal{B}^n$ for $\geqslant_{\mathcal{V}^n}$ can be obtained considering the set of weights

$$\begin{aligned} b_n(p,h) &:= \frac{(n-1)!}{h^{n-1}} \int_p^1 \left(\int_{p_1}^1 \cdots \left(\int_{p_{n-3}}^1 I_{[0,h]}(t)dt \right) \ldots dp_2 \right) dp_1 \\ &= \frac{(n-1)[(h-p)_+]^{n-2}}{h^{n-1}} \text{ where } h \in (0,1]. \end{aligned} \tag{10}$$

The bases for $n = 4$ are given by the set of all decreasing *quadratic* weighting functions censored for values below 0 and such that the lowest value is 0. For $n = 5$ *cubic* functions with similar properties are considered, and so on for higher orders.

By repeated application of integration by parts it can be shown that

$$\Delta^n(p) = \int_0^p \Delta^{n-1-k}(t) \frac{(p-t)^k}{k!} dt = \int_0^p \Delta(t) \frac{(n-1)(p-t)^{n-2}}{(n-1)!} dt \tag{11}$$

which can be rewritten as

$$\Delta^n(p) = \frac{p^{n-1}}{(n-1)!} \int_0^1 \Delta_p(t)(n-1)(1-t)^{n-2} dt. \tag{12}$$

That is

$$\Delta^n(p) = \frac{p^{n-1}}{(n-1)!} \cdot \left[\Xi(n-1; X_p) - \Xi(n-1; Y_p) \right], \tag{13}$$

note that when $p = 1$ we get as shown in Muliere and Scarsini (1990) $\Delta^n(1) = [\Xi(n-1; X) - \Xi(n-1; Y)]/(n-1)!$

Similarly applying $\geqslant_{\mathcal{B}^n}$ we obtain

$$\int_0^1 \frac{(n-1)[(h-p)_+]^{n-2}}{h^{n-1}} \cdot \Delta(t)dt = \int_0^1 (n-1)(1-t)^{n-2} \cdot \Delta_h(t)dt \geq 0$$

for all $h \in (0,1)$, which shows that the functionals $\Xi(n-1; X_p)$ for all $p \in (0,1]$ can indeed be considered bases for n^{th}–degree ISD. All results concerning stochastic orders and ISD for $n \geq 1$ can be summarized in the following proposition.

Proposition 3.4. $X \geqslant_{\mathcal{V}^n} Y \Longleftrightarrow X \geqslant_{\mathcal{B}^n} Y \Longleftrightarrow X \succcurlyeq_n Y.$

Focussing on $\geqslant_{\mathcal{B}^n}$ it is possible to restate the n^{th}–degree ISD in terms of the stochastic order induced by S-Ginis of order $n-1$ evaluated over all distributions truncated at p. That is for $n > 1$ we have

Remark 3.5. $\Xi(n-1; X_p) \geq \Xi(n-1; Y_p)$ for all $p \in [0, 1] \Longleftrightarrow X \succcurlyeq_n Y$.

Note the analogy with Foster and Shorrocks (1988) result on the relationship between stochastic dominance and the stochastic order induced by the Foster, Greer and Thorbecke (1984) [FGT] parametric family of additively decomposable poverty measures. Stochastic dominance of order n turns out to be equivalent to poverty dominance for the FGT poverty index of parameter n evaluated for any poverty line within the income domain [see also Fishburn, 1976]. Our result looks at the quantile space and instead of considering distributions censored at the poverty line focuses on their truncations at the p quantile, moreover the relevant stochastic order turns out to be the one associated with the S-Gini of order $n-1$.

3.2. *Threshold based weights*

Next set of conditions involves weighting functions characterized by their behavior conditional on a threshold level $\alpha > 0$ identifying a position in the income ranking. We investigate reasonable assumptions on the shape of the weighting functions that take explicitly into account information on a normatively significant threshold level specified in the population space. We provide novel characterization results of stochastic orders and associated dominance conditions that are in between first and second ISD or depart from second ISD without necessarily implying higher orders of ISD.

3.2.1 -*One condition in between First and Second degree dominance*

We start considering sets of non-negative weights where higher concern is given to incomes ranked in positions below the threshold $\alpha > 0$ than to incomes of individuals ranked above α. We suppose that weights for positions below α are larger than those for positions above α, but we do not assume any further restriction on comparisons of weights located on the same side of the threshold. That is we focus on the set

$$\mathcal{V}_\alpha^1 := \{v \in \mathcal{V}^1 : v(p) \geq v(q) \text{ for all } 0 \leq p \leq \alpha < q \leq 1\}.$$

Deriving the bases $\mathcal{B}_\alpha^1$ for $\geqslant_{\mathcal{V}_\alpha^1}$ turns out to be slightly cumbersome, they are identified by the set of weights $v_\alpha^1(p, \ell, \mathbf{h}, n)$ for $n \in \mathbb{N} \cup 0$ such that

$$v_\alpha^1(p, \ell, \mathbf{h}, n) := \begin{cases} \frac{1}{h_0 - \ell_0} \cdot \left[I_{[0,h_0)} - I_{[0,\ell_0)}\right] & \text{if } n = 0 \\ \frac{1}{\alpha + \sum\limits_{i=1}^{n}(h_i - \ell_i)} \cdot \left[I_{[0,\alpha)} + \sum\limits_{i=1}^{n}\left(I_{[0,h_i)} - I_{[0,\ell_i)}\right)\right] & \text{if } n \geq 1 \end{cases} \quad (14)$$

where $0 \leq \ell_0 < h_0 \leq \alpha$; and $\alpha \leq \ell_i < h_i \leq 1$ for all $i \in \{1, 2, \ldots, n\}$ with $\ell_i > h_{i-1}$ for all $i \in \{2, \ldots, n\}$. Thus

$$\mathcal{B}_\alpha^1 := \{v_\alpha^1(p, \ell, \mathbf{h}, n) \text{ for all } n \in \mathbb{N} \cup 0\},$$

that is $\mathcal{B}_\alpha^1$ considers weights assigning positive value to any income ranked below α, and once a positive weight is assigned to groups of individuals ranked above α it requires that all individuals ranked below α receive the same weight as those above α. As a result

Lemma 3.1. $X \geqslant_{\mathcal{B}_\alpha^1} Y \iff X \geqslant_{\mathcal{V}_\alpha^1} Y$.

Remark 3.6. Note that $\mathcal{B}_\alpha^1 \subset \mathcal{B}^1$ and $\mathcal{B}^2 \subset \mathcal{B}_\alpha^1$ thus

$$X \succcurlyeq_1 Y \implies X \geqslant_{\mathcal{B}_\alpha^1} Y \implies X \succcurlyeq_2 Y.$$

In order to formalize the dominance conditions associated with $\geqslant_{\mathcal{B}_\alpha^1}$ (and equivalently also with $\geqslant_{\mathcal{V}_\alpha^1}$) we need to introduce the indicator function $I_{[\Delta(p)\leq 0]}(p)$ identifying all values of p associated with a non-positive sign of $\Delta(p)$:

$$I_{[\Delta\leq 0]}(p) = \begin{cases} 1 & \text{if } \Delta(p) \leq 0 \\ 0 & \text{if } \Delta(p) > 0 \end{cases} \tag{15}$$

In next proposition we will make use of the functions $\int_\alpha^1 I_{[\Delta\leq 0]}(p)dp$ giving the average of all negative values of $\Delta(p)$ for p above the threshold α. We will also make explicit the difference between the generalized Lorenz curves of the two distributions recalling that $\Delta GL(p) := GL_X(p) - GL_Y(p) = \Delta^2(p)$. As proved in the Appendix:

Theorem 3.1. $X \geqslant_{\mathcal{V}_\alpha^1} Y$ *if and only if:*

(i) $\Delta(p) \geq 0$ *for all* $0 \leq p < \alpha$ *and*

(ii) $-\int_\alpha^1 I_{[\Delta\leq 0]}(p)\Delta(p)dp \leq \Delta GL(\alpha)$.

To summarize, condition (i) requires that for all positions below α distribution X always rank dominates distribution Y, while condition (ii) requires that if distribution Y rank dominates distribution X for positions above α then it should not be the case that the overall income advantages of Y above α are larger than the overall advantages of X below α. Of course if X rank dominates Y for all positions then dominance is obtained as argued in Remark 3.6.

The dominance condition turns out to be particularly interesting when distribution with equal means are compared. In this case dominance can be obtained applying a sequence of *quantile preserving spreads* as in Mendelson (1987), these are associated with transfers of income from individuals located above α to individuals below α. Our result requires that distribution X rank dominates distribution Y for all positions below α while Y rank dominates X for positions above α. Next corollary thus provides a new additional characterization of the dominance condition presented in Mendelson (1987).

Corollary 3.1. *If* $\mu(X) = \mu(Y)$, *then* $X \geqslant_{\mathcal{V}_\alpha^1} Y$ *if and only if:*

(i) $\Delta(p) \geq 0$ *for all* $0 \leq p < \alpha$, *and*
(ii) $\Delta(p) \leq 0$ *for all* $1 \geq p > \alpha$.

3.2.2 -Conditions implied by Second degree dominance

Within the set of inequality averse evaluations it is possible to restrict the set of admissible weighting functions imposing further restrictions on $v \in \mathcal{V}^2$. One may for instance restrict attention to weight profiles where individuals above a given relative position $\alpha > 0$ are required to exhibit weights below the average weight of societal incomes corresponding to 1. That is we can restrict attention to the set

$$\mathcal{V}_\alpha^- := \{v \in \mathcal{V}^2 : v(p) \leq 1 \text{ for all } p \geq \alpha\}.$$

The class of welfare orderings $\geqslant_{\mathcal{V}_\alpha^-}$ is consistent with the set of restrictions on the distortion functions f derived in Gajdos (2004). For discrete distributions Gajdos (2004) proves that the set $\mathcal{V}_\alpha^-$ of weighting functions is consistent with the welfare improving (inequality reducing) effect, according to the class of GG-SEFs, of transfers of incomes from an individual in position $\alpha \in (0, 1)$ uniformly spread across to all the other individuals in the distribution. Here we supplement the result in Gajdos (2004) formalizing the class of stochastic dominance conditions induced by $\geqslant_{\mathcal{V}_\alpha^-}$.

The set of bases $\mathcal{B}_\alpha^-$ for the family $\mathcal{V}_\alpha^-$ are the weights $v_\alpha^-(p, \ell, h, \alpha)$ that are conditional on the parameter $\alpha \in (0, 1]$ identifying a threshold and two extreme points $\ell \leq h$ such that $1 \geq h \geq \ell > 0$, and $\ell \leq \alpha$:

$$v_\alpha^-(p, \ell, h) := \begin{cases} [1-(h-\ell)]/\ell & \text{for } p \in [0, \ell) \\ 1 & \text{for } p \in [\ell, h) \text{ where } \ell \leq \alpha;\ \ell \leq h. \\ 0 & \text{for } p \in [h, 1] \end{cases} \quad (16)$$

That is

$$v_\alpha^-(p, \ell, h) := [1-h]/\ell \cdot I_{[0,\ell)} + I_{[0,h)} \quad \text{where } \ell \leq \alpha; \ell \leq h.$$

Lemma 3.2. $X \geqslant_{\mathcal{B}_\alpha^-} Y \iff X \geqslant_{\mathcal{V}_\alpha^-} Y$.

Application of the set of weights $\mathcal{B}_\alpha^-$ leads to a result where comparisons of convex combinations of incomplete means are required:

Theorem 3.2. *Let* $\alpha \in (0, 1)$, $X \geqslant_{\mathcal{V}_\alpha^-} Y$ *if and only if:*

(i) $\Delta GL(p) \geq 0$ *for all* $p < \alpha$, *and* $p = 1$, *and*
(ii) $\min_{0<q<\alpha}\left\{\frac{\Delta GL(q)}{q}\right\} + \min_{1>p\geq\alpha}\left\{\frac{\Delta GL(p)}{1-p}\right\} \geq 0$.

The partition of the population into two groups including individuals ranked on both sides of the relative position α will allow to make between groups comparisons and therefore will open the way to the possibility that improvements within a group lower down in the income scale can compensate for not necessarily inequality reducing changes in the situation of individuals belonging to higher income group. Note that $\Delta GL(p)/p$ measures the improvement in average income of the distributions truncated at quantile p. Condition (i) requires that generalized Lorenz dominance is satisfied for all groups of poorer individuals below the population threshold α and also requires that on average distribution X is not worse off than distribution Y. Condition (ii) can be rewritten as

$$\min_{0<q<\alpha}\left\{\frac{\Delta GL(q)}{q}\right\} \geq \max_{1>p\geq\alpha}\left\{\frac{\Delta GL(1)-\Delta GL(p)}{1-p} - \frac{\Delta\mu}{1-p}\right\}$$

where $\Delta\mu := \mu(X) - \mu(Y)$ and the term within the brackets on the right hand side is the difference between two components: it is a measure of the improvement in mean income of the richer $(1-p)$ percentage of individuals $[\Delta GL(1) - \Delta GL(p)] / (1-p)$ compared to a reference improvement $\Delta\mu/(1-p)$ that could have been obtained if the entire difference in total income between X and Y where distributed solely to the richer $(1-p)$ percentage of individuals. Condition (ii) thus requires that the minimum improvement, in moving from Y to X, among all groups of poorer individuals below α is sufficiently large to compensate for the maximal improvement between the groups of richer individuals above α whose realization exceeds the potential per capita advantage attainable sharing the surplus $\Delta\mu$. In other words richer individuals above α are allowed to improve their situation, in excess w.r.t. the advantage deriving by the sharing of $\Delta\mu$, provided that on average their exceeding improvement is not larger than the minimal average improvement taking place in groups of poor individuals below α.

When distributions have the same mean we obtain:

Corollary 3.2. *Let $\alpha \in (0,1)$, if $\mu(X) = \mu(Y)$, $X \geqslant_{\mathcal{V}_\alpha^-} Y$ if and only if:*

(i) $\Delta L(p) \geq 0$ *for all* $p < \alpha$*, and*

(ii) $\min_{0<q<\alpha}\left\{\frac{\Delta L(q)}{q}\right\} \geq \max_{1>p\geq\alpha}\left\{\frac{\Delta L(1)-\Delta L(p)}{1-p}\right\}$.

In this case since $\Delta\mu = 0$ improvements of the situation of groups of $(1-p)$ richer individuals ranked above α is admissible only if on average such improvements are not larger than those occurring to the groups of poor individuals below α.

Remark 3.7. Note that $\mathcal{B}_\alpha^- \subset \mathcal{B}^2$ thus $X \succcurlyeq_2 Y \implies X \geqslant_{\mathcal{V}_\alpha^-} Y$.

Lorenz dominance clearly implies $\geqslant_{\mathcal{V}_\alpha^-}$ since it requires that on average the of groups of $(1-p)$ richer individuals experience a worsening of their situation in moving from Y to X (i.e. $\Delta L(1) - \Delta L(p) \leq 0$ for all p).

An alternative view of taking into account the informational content of thresholds α within the set of inequality averse evaluations, may require that the weights $v \in \mathcal{V}^2$ associated with all incomes ranked below the relative position α are set above the overall population average weight corresponding to 1. Thus we consider the class of weighting functions:

$$\mathcal{V}_\alpha^+ := \{v \in \mathcal{V}^2 : v(p) \geq 1 \text{ for all } p \leq \alpha\}.$$

The base $\mathcal{B}_\alpha^+$ for $\mathcal{V}_\alpha^+$ is given by the set of

$$v_\alpha^+(p, \ell, h) := [1-h]/\ell \cdot I_{[0,\ell]} + I_{[0,h]} \quad \text{where } \alpha \leq h; \ell \leq h.$$

Readjusting Lemma 3.2 simply adapting the constraints on the values of ℓ and h it is immediate that

Lemma 3.3. $X \geqslant_{\mathcal{B}_\alpha^+} Y \iff X \geqslant_{\mathcal{V}_\alpha^+} Y$.

Applying dominance for the base $\mathcal{B}_\alpha^+$ we obtain:

Theorem 3.3. *Let* $\alpha \in (0,1)$, $X \geqslant_{\mathcal{V}_\alpha^+} Y$ *if and only if:*

(i) $\Delta GL(p) \geq 0$ *for all* $p \geq \alpha$, *and*

(ii) $\min_{\alpha > p > 0} \left\{\frac{\Delta GL(p)}{p}\right\} + \min_{1 > q \geq \alpha} \left\{\frac{\Delta GL(q)}{1-q}\right\} \geq 0.$

The above result is a modification of Theorem 3.2 since it still involves condition (ii) but combines it with the requirement of generalized Lorenz dominance for all values above α.

Combining the restrictions in each class we obtain the base $\mathcal{B}_\alpha^{+/-}$ for $\mathcal{V}_\alpha^+ \cap \mathcal{V}_\alpha^-$ that are given by the set of weights

$$v_\alpha^{+/-}(p, \ell, h) := [1-h]/\ell \cdot I_{[0,\ell]} + I_{[0,h]} \quad \text{where } 0 < \ell \leq \alpha \leq h < 1.$$

Applying the base $\mathcal{B}_\alpha^{+/-}$ we obtain:

Corollary 3.3. *Let* $\alpha \in (0,1)$, $X \geqslant_{\mathcal{V}_\alpha^+ \cap \mathcal{V}_\alpha^-} Y$ *if and only if:*

(i) $\Delta GL(\alpha) \geq 0$, *and*

(ii) $\min_{\alpha > p > 0} \left\{\frac{\Delta GL(p)}{p}\right\} + \min_{1 > q \geq \alpha} \left\{\frac{\Delta GL(q)}{1-q}\right\} \geq 0.$

While if we consider $\mathcal{V}_\alpha^+ \cup \mathcal{V}_\alpha^-$ we have:

Corollary 3.4. *Let* $\alpha \in (0,1)$, $X \geqslant_{\mathcal{V}_\alpha^+ \cup \mathcal{V}_\alpha^-} Y$ *if and only if* $X \succcurlyeq_2 Y$.

Following Castagnoli and Muliere (1990) and Mosler and Muliere (1996, 1998) one could argue that it might be reasonable to assume that progressive transfers induce a non-negative welfare effect but might exhibit a strictly positive welfare effect if they take place between a donor located above position $\alpha \in (0, 1)$ and a recipient located below α. The class of weighting functions that are consistent with this idea is denoted by $\mathcal{V}_\alpha^*$ where:

$$\mathcal{V}_\alpha^* := \{v \in \mathcal{V}^2 : v(p) > v(q) \text{ for all } p \leq \alpha < q\},$$

while the base $\mathcal{B}_\alpha^*$ for $\mathcal{V}_\alpha^*$ is given by the weighting functions

$$v_\alpha^*(p, \ell, h, \alpha) := [1 - h]/\ell \cdot I_{[0,\ell]} + I_{[0,h]} \quad \text{where } 0 < \ell \leq \alpha \leq h < 1.$$

Note however that $\overline{co}(\mathcal{V}_\alpha^*) = \mathcal{V}^2$ that is according to Remark 3.2 we have that $\geqslant_{\mathcal{V}_\alpha^*} \longleftrightarrow \geqslant_{\mathcal{V}^2}$ which implies that $\geqslant_{\mathcal{V}_\alpha^*} \longleftrightarrow \succcurlyeq_2$ with no additional restrictions on the stochastic order in moving from $\mathcal{V}^2$ to $\mathcal{V}_\alpha^*$.

4. Conclusions

Generalized Gini functionals are common in the literature on evaluation of income distributions as well as many associated dominance conditions as for instance the *inverse stochastic dominance* criteria. For a given set of distortion [weighting] functions we have identified the *maximal set* of distortion [weighting] functions that is consistent with the stochastic order induced by the former set according to the class of generalized Gini functionals. Moreover, our methodology can be used to identify also *bases* of a stochastic order, that is small sets of distortion [weighting] functions inducing a given order. Our results allow to focus attention on a small set of distortions [weights] in order to assess the normative relevance of a given order. This is precisely what we have done for the class of *inverse stochastic dominance* conditions. We have identified the relevant bases for each order of dominance and we have highlighted the equivalence between them and the stochastic orders induced by *single parameter Gini functionals* specified for a given parameter value and applied over distributions truncated at a given position in the income rank. Dominance for all such stochastic orders associated with parameter value $n - 1$ and checked irrespective of the truncation point is equivalent to inverse stochastic dominance of order n.

If additional information on normatively relevant thresholds in the ranking space is available it is then possible to identify bases that are consistent with those associated with the various degree of inverse stochastic dominance but exhibit different behavior whether applied above or below the threshold. Some stochastic orders implied by first and second order dominance have been investigated. Following similar lines of reasoning it is also possible to extend the approach deriving threshold based dominance conditions implied by higher orders of inverse stochastic dominance.

APPENDIX

If X is a random variable we call *pseudoinverse* of $\bar{F}_X$ the function defined, for all $p \in (0, 1)$, by

$$\bar{F}_X^{-1}(p) = \min\left\{t \in \mathbb{R} : \bar{F}_X(t) \leq p\right\}.$$

Next proposition collects some known properties of pseudoinverses (see e.g. Letta, 1993, and Denneberg, 1994).

Proposition A.1. *Let X, X_n, Y be random variables.*

1. *$\bar{F}_X^{-1}$ is well defined, non-increasing and right-continuous.*
2. *For all $a, b \in \mathbb{R}$ and $p \in (0, 1)$, $\bar{F}_X(a) > p \geq \bar{F}_X(b)$ if and only if $a < \bar{F}_X^{-1}(p) \leq b$.*
3. *$\bar{F}_X^{-1}$ as a random variable on $(0, 1)$ with the Borel measure has decumulative distribution function $\bar{F}_X$.*
4. *Let $g : \mathbb{R} \to \mathbb{R}$ be a Borel function and assume $g(X)$ is summable, then*
$$\int_\Omega g(X)\,dP = -\int_{\mathbb{R}} g\,d\bar{F}_X = \int_0^1 g\left(\bar{F}_X^{-1}(t)\right)dt.$$
5. *$X_n \Rightarrow X$ if and only if $\bar{F}_{X_n}^{-1}(p)$ converges to $\bar{F}_X^{-1}(p)$ for all $p \in (0, 1)$ at which $\bar{F}_X^{-1}$ is continuous.*
6. *$\bar{F}_X \geq \bar{F}_Y$ if and only if $\bar{F}_X^{-1} \geq \bar{F}_Y^{-1}$.*
7. *If $a \geq 0$ and $b \in \mathbb{R}$, then $\bar{F}_{aX+b}^{-1} = a\bar{F}_X^{-1} + b$.*
8. *If X and Y are comonotonic, $\bar{F}_{X+Y}^{-1} = \bar{F}_X^{-1} + \bar{F}_Y^{-1}$.*

Proof of Theorem 2.1. The proof follows Maccheroni (2004). By Eq. (1) it is clear that

$$X \geqslant_{\mathcal{F}} Y \Leftrightarrow \int_{\Omega} Xd\,(u \circ P) \geq \int_{\Omega} Yd\,(u \circ P)\,, \quad \forall u \in \overline{co}\,(\mathcal{F})\,.$$

Assume by contradiction that $\overline{co}\,(\mathcal{F})$ is not the maximal set of distortions representing $\geqslant_{\mathcal{F}}$, then there exists a distortion $g \notin \overline{co}\,(\mathcal{F})$ such that

$$X \geqslant_{\mathcal{F}} Y \Rightarrow \int_{\Omega} Xd\,(g \circ P) \geq \int_{\Omega} Yd\,(g \circ P)\,. \tag{17}$$

Let $\mathcal{X}$ be the set of all random variables taking values in [0, 1]. First notice that for all $X \in \mathcal{X}$, $\int_{\Omega} Xd\,(f \circ P) = \int_0^1 f\left(\bar{F}_X(t)\right) dt$. Hence, $\geqslant_{\mathcal{F}}$ can be regarded as a preorder on the family of the decumulative distribution functions of all random variables in $\mathcal{X}$, or – more precisely – on the set of Γ of their restrictions to [0, 1]. By the nonatomicity of P, Γ is the set of all non-increasing, right-continuous functions $J : [0,1] \to [0,1]$ such that $J(1) = 0$, with the convention $J\left(0^-\right) = 1$.

For all $X \in \mathcal{X}$, we can extend $\bar{F}_X^{-1}$ to the whole [0, 1] by setting $\bar{F}_X^{-1}(0) = \min\left\{t \in \mathbb{R} : \bar{F}_X(t) \leq 0\right\}$ and $\bar{F}_X^{-1}(1) = 0$. Given $J = \left(\bar{F}_X\right)_{|[0,1]} \in \Gamma$, by J^{-1} we mean $\bar{F}_X^{-1}$. For all $J \in \Gamma$, the following properties hold:

- $J^{-1}(p) = \min\{t \in [0,1] : J(t) \leq p\} = \min\left\{t \in [0,1] : J(t) \leq p \leq J(t^-)\right\}$,
- $J^{-1} \in \Gamma$,
- $\left(J^{-1}\right)^{-1} = J$.

The vector space generated by Γ is $RBV_1([0,1])$,([6]) which is (isomorphic to) the topological dual of $C([0,1])$, the duality being

$$\langle f, F \rangle = -\int_{[0,1]} f dF.$$

Next we show that the cone $\mathcal{H}$ generated by $\overline{co}\,(\mathcal{F})$ is closed, convex, does not contain g and $h(0) = 0$ for all $h \in \mathcal{H}$. If $\alpha u, \beta w \in \mathcal{H}$ (with $\alpha, \beta \in \mathbb{R}_+$ and $u, w \in \overline{co}\,(\mathcal{F})$) either $\alpha = \beta = 0$ and $\alpha u + \beta w = 0 \in \mathcal{H}$, or $\alpha u + \beta w = (\alpha + \beta)\left(\frac{\alpha}{\alpha+\beta} u + \frac{\beta}{\alpha+\beta} w\right) \in \mathcal{H}$, hence $\mathcal{H}$ is convex. If $\alpha_n u_n \in \mathcal{H}$ (with $\alpha_n \in \mathbb{R}_+$ and $u_n \in \overline{co}\,(\mathcal{F})$) and $\alpha_n u_n \to h$, then $\alpha_n = \alpha_n u_n(1) \to h(1)$; therefore, either $h(1) = 0$ and $\alpha_n u_n \to 0 \in \mathcal{H}$, or $u_n = \frac{\alpha_n u_n}{\alpha_n} \to \frac{h}{h(1)} \in \overline{co}\,(\mathcal{F})$ and $h \in \mathcal{H}$, hence $\mathcal{H}$ is closed. If $g \in \mathcal{H}$, then $g = \alpha u$ with $\alpha \in \mathbb{R}_+$ and

([6]) RBV_1 denotes the set of all functions $\phi : \left[0^-, 1\right] \to \mathbb{R}$ such that ϕ is of bounded variation, right continuous, and $\phi(1) = 0$.

$u \in \overline{co}(\mathcal{F})$, in particular $1 = g(1) = \alpha \cdot u(1) = \alpha$, and $g \in \overline{co}(\mathcal{F})$, a contradiction.

The set $\{h + \alpha : h \in \mathcal{H} \text{ and } \alpha \in \mathbb{R}\}$ is a convex cone too. Let $\mathcal{K}$ denote its closure, if $g \in \mathcal{K}$ there exist sequences $h_n \in \mathcal{H}$ and $\alpha_n \in \mathbb{R}$ s.t. $h_n + \alpha_n \to g$. Hence $h_n(0) + \alpha_n \to g(0)$, but $h_n(0) = g(0) = 0$, consequently $\alpha_n \to 0$, and $h_n = (h_n + \alpha_n) - \alpha_n \to g$. So $g \in \overline{\mathcal{H}} = \mathcal{H}$, which is absurd. By the Separating Hyperplane Theorem (see, e.g., Aliprantis and Border, 1999, Theorem 5.58) there exists a nonzero function $\phi \in RBV_1([0,1])$ such that

$$-\int_{[0,1]} g d\phi > 0 \geq -\int_{[0,1]} k d\phi, \quad \forall k \in \mathcal{K}.$$

Since the constant functions belong to $\mathcal{K}$, then $\alpha\phi(0^-) = -\int_{[0,1]} \alpha d\phi \leq 0$ for all $\alpha \in \mathbb{R}$, and $\phi(0^-) = 0$. Therefore there exist $J_1, J_2 \in \Gamma$ and $\gamma > 0$ such that $\phi = \gamma(J_1 - J_2)$, so

$$-\gamma \int_{[0,1]} g d(J_1 - J_2) > 0 \geq -\gamma \int_{[0,1]} k d(J_1 - J_2), \quad \forall k \in \mathcal{K}.$$

For $i = 1, 2$, set $H_i = J_i^{-1} \in \Gamma$, since $\mathcal{F} \subseteq \mathcal{K}$, we have $-\int_{[0,1]} f d(H_1^{-1} - H_2^{-1}) \leq 0$ for all $f \in \mathcal{F}$, that is

$$\int_0^1 f(H_2(t))\, dt = -\int_{[0,1]} f dH_2^{-1} \geq -\int_{[0,1]} f dH_1^{-1} = \int_0^1 f(H_1(t)) dt \;\; \forall f \in \mathcal{F}$$

and $H_2 \geqslant_{\mathcal{F}} H_1$. But $-\int_{[0,1]} g d\left(H_1^{-1} - H_2^{-1}\right) > 0$, that is $\int_0^1 g(H_1(t))\, dt > \int_0^1 g(H_2(t))\, dt$, which is absurd. □

Proof of Corollary 2.1. By Theorem 2.1 if $\overline{co}(\mathcal{G}) \subseteq \overline{co}(\mathcal{F})$, then

$$X \geqslant_{\mathcal{F}} Y \Leftrightarrow X \geqslant_{\overline{co}(\mathcal{F})} Y \Rightarrow X \geqslant_{\overline{co}(\mathcal{G})} Y \Leftrightarrow X \geqslant_{\mathcal{G}} Y,$$

for all $X, Y \in \mathcal{L}_\infty^+$. Conversely, if $X \geqslant_{\mathcal{F}} Y \Rightarrow X \geqslant_{\mathcal{G}} Y$ and there exists $g \in \overline{co}(\mathcal{G}) \backslash \overline{co}(\mathcal{F})$, by the separation argument used in the proof of Theorem 2.1, there exist $X_1, X_2 \in \mathcal{L}_\infty^+$ such that $X_2 \geqslant_{\mathcal{F}} X_1$ but $\int_\Omega X_1 d(g \circ P) > \int_\Omega X_2 d(g \circ P)$. Therefore it cannot be $X_2 \geqslant_{\mathcal{G}} X_1$, which is absurd. □

Proof of Proposition 2.1. First notice that $co(\mathcal{V}) \subseteq \widehat{co}(\mathcal{V})$. Let $v_1, \ldots, v_n \in \mathcal{V}$ and $\alpha_1, \ldots, \alpha_n \geq 0$ with $\sum_1^n \alpha_i = 1$, then

$$f_{\sum_1^n \alpha_i v_i}(p) = \int_0^p \sum_1^n \alpha_i v_i (1-t)\, dt = \sum_1^n \alpha_i f_{v_i}(p) \quad \forall p \in [0,1] \text{ and}$$

$$g := f_{\sum_1^n \alpha_i v_i} = \sum_1^n \alpha_i f_{v_i} \in \overline{co}(\{f_v : v \in \mathcal{V}\}) \cap AC([0,1]),$$

moreover, $g'(t) = \sum_1^n \alpha_i v_i (1-t)$ and $\sum_1^n \alpha_i v_i (t) = g'(1-t) \in \widehat{co}(\mathcal{V})$. In particular, $\mathcal{V} \subseteq \widehat{co}(\mathcal{V})$.

Then notice that $\mathcal{F}_{\widehat{co}(\mathcal{V})} = \overline{co}(\{f_v : v \in \mathcal{V}\}) \cap AC([0,1])$, in fact

$$\begin{aligned} g \in \mathcal{F}_{\widehat{co}(\mathcal{V})} &\Leftrightarrow g(p) = \int_0^p f'(t)dt \text{ for some } f \in \overline{co}(\{f_v : v \in \mathcal{V}\}) \cap AC([0,1]) \\ &\Leftrightarrow g \in \overline{co}(\{f_v : v \in \mathcal{V}\}) \cap AC([0,1]). \end{aligned}$$

If $\geqslant_{\mathcal{F}_\mathcal{W}}$ is weaker than $\geqslant_{\mathcal{F}_\mathcal{V}}$ then $\overline{co}(\mathcal{F}_\mathcal{W}) \subseteq \overline{co}(\mathcal{F}_\mathcal{V})$ and

$$\overline{co}(\mathcal{F}_\mathcal{W}) \cap AC([0,1]) \subseteq \overline{co}(\mathcal{F}_\mathcal{V}) \cap AC([0,1]),$$

whence $\mathcal{W} \subseteq \widehat{co}(\mathcal{W}) \subseteq \widehat{co}(\mathcal{V})$. Conversely, if $\mathcal{W} \subseteq \widehat{co}(\mathcal{V})$, then

$$\begin{aligned} \mathcal{F}_\mathcal{W} \subseteq \mathcal{F}_{\widehat{co}(\mathcal{V})} &= \overline{co}(\{f_v : v \in \mathcal{V}\}) \cap AC([0,1]) \\ &= \overline{co}(\mathcal{F}_\mathcal{V}) \cap AC([0,1]) \subseteq \overline{co}(\mathcal{F}_\mathcal{V}), \end{aligned}$$

and hence $\overline{co}(\mathcal{F}_\mathcal{W}) \subseteq \overline{co}(\mathcal{F}_\mathcal{V})$, which in turn implies that $\geqslant_\mathcal{W}$ is weaker than $\geqslant_\mathcal{V}$.

1. $\mathcal{V} \subseteq \widehat{co}(\mathcal{V})$ implies that $\geqslant_\mathcal{V}$ is weaker than $\geqslant_{\widehat{co}(\mathcal{V})}$. Conversely, $\geqslant_{\widehat{co}(\mathcal{V})}$ is, by definition, $\geqslant_{\mathcal{F}_{\widehat{co}(\mathcal{V})}}$ which, by $\mathcal{F}_{\widehat{co}(\mathcal{V})} \subseteq \overline{co}(\mathcal{F}_\mathcal{V})$, is weaker than $\geqslant_{\overline{co}(\mathcal{F}_\mathcal{V})} = \geqslant_{\mathcal{F}_\mathcal{V}} = \geqslant_\mathcal{V}$. We conclude that $\geqslant_{\widehat{co}(\mathcal{V})} = \geqslant_\mathcal{V}$. Moreover, if $\geqslant_\mathcal{W}$ coincides with $\geqslant_\mathcal{V}$, then $\geqslant_\mathcal{W}$ is weaker than $\geqslant_\mathcal{V}$ and $\mathcal{W} \subseteq \widehat{co}(\mathcal{V})$.
2. $\geqslant_\mathcal{V}$ and $\geqslant_\mathcal{W}$ coincide if and only if $\geqslant_{\widehat{co}(\mathcal{V})}$ is weaker than $\geqslant_\mathcal{W}$ so that $\widehat{co}(\mathcal{V}) \subseteq \widehat{co}(\mathcal{W})$, and $\geqslant_{\widehat{co}(\mathcal{W})}$ is weaker than $\geqslant_\mathcal{V}$, so that $\widehat{co}(\mathcal{W}) \subseteq \widehat{co}(\mathcal{V})$. □

Proof of Corollary 2.2. We first show that $\overline{co}^{\mathcal{L}_1}(\mathcal{B}) \subseteq \widehat{co}(\mathcal{B})$. If $v \in \overline{co}^{\mathcal{L}_1}(\mathcal{B})$, then there exists a sequence v_n in $co(\mathcal{B})$ that converges to v in $\mathcal{L}_1([0,1])$. Therefore

$$\sup_{A \text{ Borel}} \left| \int_A v^n(t)\, dt - \int_A v(t)\, dt \right| \to 0$$

since $\mathcal{L}_1([0,1])$ is norm isometric to the space of all signed measures which are absolutely continuous w.r.t. the Lebesgue measure, endowed with the total variation norm, which – in turn – is equivalent to the supnorm (on that space, see, e.g. Dunford and Schwartz, 1958). In particular $\int_{1-p}^{1} v^n(t)\,dt$ uniformly converges to $\int_{1-p}^{1} v(t)\,dt$ therefore f_v belongs to the closure of $co(\mathcal{F}_{\mathcal{B}})$.

Since $\mathcal{B} \subseteq \mathcal{V}$, then $\geqslant_{\mathcal{B}}$ is weaker than $\geqslant_{\mathcal{V}}$. Since for every $v \in \mathcal{V}$ there exists a sequence v_n in $co(\mathcal{B})$ that converges to v in $\mathcal{L}_1([0,1])$, then $v \in \overline{co}^{\mathcal{L}_1}(\mathcal{B}) \subseteq \widehat{co}(\mathcal{B})$. Thus $\mathcal{V} \subseteq \widehat{co}(\mathcal{B})$ and $\geqslant_{\mathcal{V}}$ is weaker than $\geqslant_{\mathcal{B}}$. □

Lemma A.1. $X \geqslant_{\mathcal{V}^1} Y \iff X \geqslant_{\mathcal{B}^1} Y$.

Proof. For any $v \in \mathcal{V}^1$ we have that $v \in \mathcal{L}_1^+([0,1])$ and $\int_0^1 v(t)dt = 1$. Let

$$\pi_n := \left\{\left[0, \frac{1}{2^n}\right); \left[\frac{1}{2^n}, \frac{2}{2^n}\right); \dots; \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right); \dots; \left[\frac{2^n-1}{2^n}, \frac{2^n}{2^n}\right)\right\} \quad (18)$$

for all $n \in \mathbb{N}$ (set of natural numbers) denote a sequence of partitions of $[0,1]$ into 2^n intervals of equal length. Denote by $v^n \equiv E(v|\pi_n)$ the conditional expectation of v given partition π_n. That is, given

$$v^n(p) := 2^n \cdot \int_{\frac{j}{2^n}}^{\frac{j+1}{2^n}} v(t)dt = 2^n \cdot c_j^n \text{ if } p \in [\frac{j}{2^n}, \frac{j+1}{2^n}) \quad (19)$$

we have that

$$v^n = 2^n \cdot \sum_{j=0}^{2^n-1} c_j^n \cdot I_{[\frac{j}{2^n}, \frac{j+1}{2^n})}. \quad (20)$$

Recalling that $I_{[\frac{j}{2^n}, \frac{j+1}{2^n})} = I_{[0, \frac{j+1}{2^n})} - I_{[0, \frac{j}{2^n})}$ we have that $2^n \cdot I_{[\frac{j}{2^n}, \frac{j+1}{2^n})}$ belongs to $\mathcal{B}^1$ according to the definition in (5). Therefore v^n is a convex combination of elements in $\mathcal{B}^1$. It is well known (see Royden 1987, p.129, Proposition 9) that for any normalized $v \in \mathcal{V}^1$ the constructed v^n in (19) [and therefore also in (20)] converges to v, which proves that $\mathcal{V}^1 \subseteq \overline{co}(\mathcal{B}^1)$. It follows that $\overline{co}(\mathcal{V}^1) \subseteq \overline{co}(\mathcal{B}^1)$ which together with the fact that $\mathcal{B}^1 \subseteq \mathcal{V}^1$, according to Remark 3.2, leads to the required result. □

Lemma A.2. $X \geqslant_{\mathcal{V}^2} Y \iff X \geqslant_{\mathcal{B}^2} Y$.

Proof. For any $v \in \mathcal{V}^2$ then $v \in \mathcal{L}_1^+([0,1])$, $\int_0^1 v(t)dt = 1$, and v is non-increasing. Denote by $v^n \equiv E(v|\pi_n)$ the conditional expectation of v given partition π_n as in (18). That is, given (19) we have $v^n = 2^n \cdot \sum_{j=0}^{2^n-1} c_j^n \cdot I_{[\frac{j}{2^n}, \frac{j+1}{2^n})}$ where $c_j^n := \int_{\frac{j}{2^n}}^{\frac{j+1}{2^n}} v(t)dt$ and v^n is non-increasing since $c_j^n \geq c_{j+1}^n$. Making use of Abel's lemma it follows that $v^n = 2^n \cdot \sum_{j=0}^{2^n-1} (c_j^n - c_{j+1}^n) \cdot I_{[0, \frac{j+1}{2^n})}$ where

$c^n_{2^n} := 0$. Note that letting $w^n_j := c^n_j - c^n_{j+1} \geq 0$ we have that $v^n = \sum_{j=0}^{2^n-1} (j+1) \cdot w^n_j \cdot [\frac{2^n}{j+1} \cdot I_{[0,\frac{j+1}{2^n})}]$ where $\frac{2^n}{j+1} \cdot I_{[0,\frac{j+1}{2^n})}$ are elements of $\mathcal{B}^2$ as in (6), while their weights $(j+1) \cdot w^n_j \geq 0$ sum to 1 given that $\sum_{j=0}^{2^n-1} (j+1) \cdot w^n_j = \sum_{j=0}^{2^n-1} [\sum_{i=j}^{2^n-1} w^n_i] = \sum_{j=0}^{2^n-1} \left(c^n_j - c^n_{2^n}\right) = \sum_{j=0}^{2^n-1} c^n_j = \int_0^1 v(t)dt = 1$. Thus v^n is a convex combination of elements in $\mathcal{B}^2$. The proof is completed replicating the same arguments applied in the final part of the proof of Lemma A.1 in order to show that $\overline{co}\left(\mathcal{V}^2\right) \subseteq \overline{co}\left(\mathcal{B}^2\right)$. □

Lemma A.3. $X \geqslant_{\mathcal{V}^3} Y \Longleftrightarrow X \geqslant_{\mathcal{B}^3} Y$.

Proof. Note that $v \in \mathcal{V}^3$ if and only if $v \in \mathcal{V}^{*2} := \{v : v \geq 0,\ v$ is non-increasing, $v(1) = 0\}$, $-v'_- \in \mathcal{V}^{*2}$ and $\int_0^1 v(t)dt = 1$. Let $b'_{3-}(p,h)$ denote the left-hand side partial derivative of $b_3(p,h)$ in (8) w.r.t. p, thus $-b'_{3-}(p,h) := \frac{2}{h^2} I_{[0,h)}$.

Take as primitive concept the function $u := -v'_-$ where by construction taking $u \in \mathcal{L}_1([0,1])$ and following the same procedure applied in the proof of Lemma A.2 it follows that

$$u^n = 2^n \cdot \sum_{j=0}^{2^n-1} c^n_j \cdot I_{[\frac{j}{2^n},\frac{j+1}{2^n})} = \sum_{j=0}^{2^n-1} \frac{(j+1)^2}{2^{n+1}} \cdot w^n_j \cdot \left[2\left(\frac{2^n}{j+1}\right)^2 \cdot I_{[0,\frac{j+1}{2^n})}\right] \tag{21}$$

where $c^n_j = \int_{\frac{j}{2^n}}^{\frac{j+1}{2^n}} v(t)dt - \int_{\frac{j+1}{2^n}}^{\frac{j+2}{2^n}} v(t)dt = v(j/2^n) - v((j+1)/2^n)$ and $w^n_j := c^n_j - c^n_{j+1} \geq 0$. Note that the term within square brackets in (21) is $-b'_{3-}(p, \frac{j+1}{2^n})$. Therefore u^n is a *linear positive* combination of elements in $\mathcal{B}^3$. As shown in Lemma A.2 we have that u^n converges to $u := -v'_-$ in $\mathcal{L}_1$. It can be shown that if $u^n \to u$ in $\mathcal{L}_1$ then $v^n(p) = \int_p^1 u^n(t)dt \to \int_p^1 u(t)dt = v(p)$ uniformly w.r.t. p. Therefore $v^n \to v$ in $\mathcal{L}_1$. In order to prove he result we need to show that if u^n is a linear positive combination of elements in $\mathcal{B}^3$ then $\mathcal{V}^3 \subseteq \overline{co}(\mathcal{B}^3)$. Note that $v^n \to v \in \mathcal{V}^3$ and that v belongs to the *closure of the cone* generated by $\overline{co}(\mathcal{B}^3)$. It can be shown that normalization of v and elements of $\mathcal{B}^3$ implies that $v \in \overline{co}(\mathcal{B}^3)$. □

Lemma A.4. $X \geqslant_{\mathcal{V}^n} Y \Longleftrightarrow X \geqslant_{\mathcal{B}^n} Y$.

The result can be completed by induction following the proof of Lemma A.3 and recalling the following facts:

(i) if $-v'_- \in \mathcal{V}^{*n-1}$ then it is possible to construct an analogous of (21) obtained from $-v'_-$ which is a positive *linear* combination of elements in $\mathcal{B}^n$,

(ii) $\mathcal{L}_1$ convergence of derivatives implies $\mathcal{L}_1$ convergence of primitives,

(iii) normalized elements of the *closure of the cone* generated by $\overline{co}(\mathcal{B}^n)$ belong to $\overline{co}(\mathcal{B}^n)$. □

Proof of Lemma 3.1. Consider a partition of [0, 1] into two intervals identified by $\alpha \in (0, 1)$. Making use of $v_\alpha^1(p, \ell, \mathbf{h}, 0)$ in (14) following the proof of Lemma A.1 we obtain positive *linear transformations* of any $v \in \mathcal{V}^1$ in $(0, \alpha)$. Note that by construction any step function belonging to $\mathcal{V}_\alpha^1$ with constant values in $(0, \alpha)$ can be constructed as a *positive linear combination* of functions $v_\alpha^1(p, \ell, \mathbf{h}, n)$ for $n \geq 1$ belonging to $\mathcal{B}_\alpha^1$. By Lemma A.1 we get that the sequence of these step functions converges to functions in $\mathcal{V}_\alpha^1$ that are constant in $(0, \alpha)$. Combining these functions with those derived making use of the bases $v_\alpha^1(p, \ell, \mathbf{h}, 0)$ we obtain the entire class $\mathcal{V}_\alpha^1$. □

Proof of Theorem 3.1. Consider Lemma 3.1 where $\geqslant_{\mathcal{V}_\alpha^1} \Leftrightarrow \geqslant_{\mathcal{B}_\alpha^1}$. Applying $\geqslant_{\mathcal{B}_\alpha^1}$ we obtain:

$$\int_0^1 v_\alpha^1(p, \ell, \mathbf{h}, n) \cdot \Delta(p) dp \geq 0 \text{ for all } n \in \mathbb{N} \cup 0. \tag{22}$$

Letting $n = 0$, we get $[\Delta GL(h_0) - \Delta GL(\ell_0)] / (h_0 - \ell_0) \geq 0$ for all $0 \leq \ell_0 < h_0 \leq \alpha$; that is denoting by $\Delta GL(h_0) - \Delta GL(\ell_0) = \int_{\ell_0}^{h_0} \Delta(t) dt$ we can rewrite

$$\frac{1}{h_0 - \ell_0} \int_{\ell_0}^{h_0} \Delta(t) dt \geq 0 \text{ for all } 0 \leq \ell_0 < h_0 \leq \alpha. \tag{23}$$

Thus, letting $h_0 \to \ell_0$ we obtain $\Delta(p) \geq 0$ for all $0 \leq p < \alpha$.

Consider a fixed $n \in \mathbb{N}$ then (2.1) is satisfied if and only if

$$\Delta GL(\alpha) + \textstyle\sum_{i=1}^n [\Delta GL(h_i) - \Delta GL(\ell_i)] \geq 0 \tag{24}$$

for all $\alpha \leq \ell_i < h_i \leq 1$ for all $i \in \{1, 2, \ldots, n\}$ with $\ell_i > h_{i-1}$ for all $i \in \{2, \ldots, n\}$.

Since $\Delta(p) \geq 0$ for all $0 \leq p < \alpha$ then $\Delta GL(\alpha) \geq 0$, let $n = 1$, $h_1 = 1$, and $\ell_1 = \alpha$, necessarily it is required that $\Delta GL(1) \geq 0$.

Condition (24) can also be rewritten as:

$$\Delta GL(\alpha) \geq -\min_{n \in \mathbb{N}} \left\{ \textstyle\sum_{i=1}^n \int_{\ell_i}^{h_i} \Delta(t) dt \right\} \tag{25}$$

with previously specified restrictions for ℓ_i and h_i. Consider the indicator function $I_{[\Delta \leq 0]}(p)$ as in (15), then the right hand side of (25) coincides with $-\int_\alpha^1 I_{[\Delta \leq 0]}(p) \Delta(p) dp$. Thus the required condition is

$$\Delta GL(\alpha) + \int_\alpha^1 I_{[\Delta \leq 0]}(p) \Delta(p) dp \geq 0.$$

Otherwise it will be always possible to find a sequence of ℓ_i and h_i such that (24) is violated. □

Proof of Corollary 3.1. If $\mu(X) = \mu(Y)$, then $\Delta GL(1) = 0$, thus $\int_\alpha^1 \Delta(p)dp = -\Delta GL(\alpha) \leq 0$.
Substituting into condition (ii) in Theorem 3.1 we get $\int_\alpha^1 I_{[\Delta \leq 0]}(p)\Delta(p)dp \geq \int_\alpha^1 \Delta(p)dp$ where both sides are non-positive, moreover by construction $\int_\alpha^1 I_{[\Delta \leq 0]} \cdot \Delta dp \leq \int_\alpha^1 \Delta dp$. These conditions are only satisfied if $\int_\alpha^1 I_{[\Delta \leq 0]}(p)\Delta(p)dp = \int_\alpha^1 \Delta(p)dp$, therefore it follows that $\Delta(p) \leq 0$ for all $1 \geq p > \alpha$. □

Proof of Lemma 3.2. Consider a partition of [0, 1] into two intervals identified by $\alpha \in (0, 1)$. According to $v_\alpha^-(p, \ell, h) := [1 - h]/\ell \cdot I_{[0,\ell)} + I_{[0,h)}$ where $\ell \leq \alpha$ and $\ell \leq h$, letting $h \to \ell$, following the proof of Lemma A.2, it is possible to obtain positive linear transformations of any $v \in \mathcal{V}^2$ in $(0, \alpha)$ taking value 0 for all elements in $(\alpha, 1)$. Letting $\ell = \alpha$ and considering $\alpha \leq h$ we obtain elements of $\mathcal{V}^2$ where by construction the value of v never exceed 1 in $(\alpha, 1)$ and is at least 1 in $(0, \alpha)$. Combining these separate results we derive $\mathcal{V}_\alpha^-$. □

Proof of Theorem 3.2. Making use of Lemma 3.2 we consider directly welfare dominance according to (16) which requires that:

$$\int_0^1 v_\alpha^-(p, \ell, h) \cdot \Delta(p)dp \geq 0 \text{ for all } \ell \leq \alpha;\ \ \ell \leq h \text{ for a given } \alpha > 0.$$

Thus

$$[1 - (h - \ell)]/\ell \cdot \int_0^\ell \Delta(p)dp + \int_\ell^h \Delta(p)dp \geq 0 \text{ for all } \ell < \alpha;\ \ \ell \leq h,$$

that is

$$[1 - (h - \ell)]/\ell \cdot \Delta GL(\ell) + \Delta GL(h) - \Delta GL(\ell) \geq 0 \text{ for all } \ell < \alpha;\ \ \ell \leq h.$$

Rearranging we obtain

$$[1 - h] \cdot \Delta GL(\ell) + \ell \cdot \Delta GL(h) \geq 0 \text{ for all } \ell < \alpha;\ \ \ell \leq h. \tag{26}$$

For a fixed value of ℓ letting $h = \ell$ we get

$$\Delta GL(\ell) \geq 0 \text{ for all } \ell < \alpha. \tag{27}$$

If we consider $h > \ell$ we have that: (i) if $h < \alpha$ condition (27) is sufficient to guarantee that (26) is satisfied, while (ii) if $h \geq \alpha$ then condition

$$\frac{\Delta GL(\ell)}{\ell} + \frac{\Delta GL(h)}{[1 - h]} \geq 0 \text{ for all } \ell < \alpha;\ \ \alpha \leq h \tag{28}$$

has to hold if $h < 1$. While if $h = 1$ we have $\Delta GL(1) \geq 0$. For a fixed value of $h \geq \alpha$, applying (27) to the first term of (28) a necessary condition for dominance for all $h \geq \alpha$ is that

$$\min_{0<\ell<\alpha} \left\{ \frac{\Delta GL(\ell)}{\ell} \right\} + \frac{\Delta GL(h)}{1-h} \geq 0 \text{ for all } \alpha \leq h.$$

Fixing α and taking all $h \geq \alpha$, then the previous condition requires that

$$\min_{0<\ell<\alpha} \left\{ \frac{\Delta GL(\ell)}{\ell} \right\} + \min_{1>h\geq\alpha} \left\{ \frac{\Delta GL(h)}{1-h} \right\} \geq 0. \tag{29}$$

□

Proof of Theorem 3.3. We make use of the fact that $X \geqslant_{\mathcal{B}_\alpha^+} Y \iff X \geqslant_{\mathcal{V}_\alpha^+} Y$ which can be proven readjusting Lemma 3.2. Welfare dominance according to $v_\alpha^+(p, \ell, h)$ in $\mathcal{B}_\alpha^+$ requires that

$$[1-h] \cdot \Delta GL(\ell) + \ell \cdot \Delta GL(h) \geq 0 \text{ for all } \alpha < h;\ \ell \leq h. \tag{30}$$

For a fixed value of $h \geq \alpha$ letting $h = \ell$ we get

$$\Delta GL(h) \geq 0 \text{ for all } h \geq \alpha. \tag{31}$$

If we consider $h > \ell$ we have that: (i) if $\ell \geq \alpha$ condition (31) is sufficient to guarantee that (30) is satisfied, while (ii) if $\ell < \alpha$ then condition

$$\frac{\Delta GL(\ell)}{\ell} + \frac{\Delta GL(h)}{1-h} \geq 0 \text{ for all } \ell < \alpha;\ \alpha \leq h \tag{32}$$

has to hold. For a fixed value of $\ell < \alpha$, applying (31) to the second term of (32) a necessary condition for dominance for all $\ell < \alpha$ is that

$$\frac{\Delta GL(\ell)}{\ell} + \min_{1>h\geq\alpha} \left\{ \frac{\Delta GL(h)}{1-h} \right\} \geq 0 \text{ for all } \ell < \alpha.$$

that is we have (29). □

ACKNOWLEDGMENTS

We are grateful to Massimo Marinacci for helpful comments and suggestions.

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