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Scale-invariant test of normality based on Polya's characterization

DIPARTIMENTO DI STATISTICA, PROBABILITA' E STATISTICHE APPLICATE
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1. INTRODUCTION

In last years there exists a growing interest to the design and to the analysis of tests based on characterizations. Typical examples are the testing of symmetry (Baringhaus and Henze (1992)) or testing exponentiality based on the lack-of-memory property (Koul (1978), Angus (1982), Ahmad and Alwasel (1999)).

However despite a large set of various characterizations of the normal law (see for example Mathai and Pederzoli (1977), Bryc 1995)) there exist very few tests of normality based on characterizations. The aim of this note is to construct the scale-invariant test of normality which is based on the celebrated Polya's characterization of normality and to discuss its asymptotic properties.

2. THE NEW TEST OF NORMALITY

Polya (1923) discovered the following elegant characterization of normality which was the first result in this direction (see Kagan, Linnik and Rao (1973) or Bryc (1995)).

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Suppose that X and Y are independent identically distributed random variables with zero mean. The equality in distribution of the random variables X and $(X+Y)/\sqrt{2}$ takes place if and only if X and Y are normally distributed.

Let $X_1, \dots, X_n$ be a sample of size n from a continuous distribution function (d.f.) with known mean (without loss of generality we may assume it is zero).The assumption of known mean is quite natural in many physical and technical models.

We want to test the hypothesis H_0 that the sample is from a normal distribution with zero mean and unknown scale parameter $\sigma > 0$. As the alternative H_1 to the hypothesis H_0 we can consider the normal distribution with nonzero mean, the skew normal distribution (see Azzalini (1985)) and, more generally, other asymmetric distributions or classes of them.

Let

$$F_n(t) = n^{-1} \sum_{i=1}^{n} I\{X_i \le t\}, t \in R^1$$

be the ordinary empirical d.f. based on the sample $X_1, \dots, X_n$. To estimate the d.f. of the random variable $(X+Y)/\sqrt{2}$ we will use the so-called V-statistical empirical d.f. given by

$$G_n(t) = n^{-2} \sum_{i,j=1}^{n} I\{X_i + X_j \le t\sqrt{2}\}, \ \ t \in R^1 .$$

The U-statistical empirical d.f. is defined in a similar way, one should only take the average over different values of indices and replace n^{-2} by $\binom{n}{2}^{-1}$.

See Korolyuk and Borovskikh (1994) or Janssen (1988) for discussion of properties of such d.f.'s. Using the version of Glivenko-Cantelli theorem for U- and V-statistical d.f.'s (see, e.g., Janssen (1988)) we may conclude that almost surely under the hypothesis of normality one has

$$\sup_{t \in R^1} |F_n(t) - G_n(t)| \to 0, \ n \to \infty .$$

Hence to test H_0 we can construct the test statistics on some "distance" between G_n and F_n. We will use the scale-free statistic

$$L_n = \int_{-\infty}^{+\infty} [F_n(t) - G_n(t)] dF_n(t)$$

motivated by the well-known statistic for testing exponentiality proposed by Hollander and Proshan (1972). It will be proved that L_n is asymptotically normal under H_0. Moreover, we will point the sufficient condition for consistency of corresponding test and will explore its efficiency for simplest alternatives.

3. Properties of test statistic

In this section we discuss some properties of the proposed test. Integrating with respect to F_n we get the almost sure representation of L_n in the form of V-statistic or von Mises functional

$$L_n = n^{-3} \sum_{i,j,k=1}^{n} \left[\frac{1}{2} - I\{X_i + X_j < X_k\sqrt{2}\}\right]$$

with the kernel Ψ of degree 3 defined for any $x, y, z \in R^1$ by the formula

$$\Psi(x,y,z)=\frac{1}{3}\left(\frac{3}{2}-I\{x+y<z\sqrt{2}\}-I\{x+z<y\sqrt{2}\}-I\{y+z < x\sqrt{2}\}\right).$$

As the kernel Ψ is bounded it is almost trivial (see, e.g., similar arguments in Eichselbacher and Loewe (1995), p. 814-815) that this statistic has identical limiting properties with the corresponding U-statistic having the same kernel, namely

$$M_n = \binom{n}{3}^{-1} \sum_{1\le i<j<k\le n}^{n} \Psi(X_i, X_j, X_k),$$

where the calculations are somewhat easier.

According to the general theory of U-statistics (see, e.g. Serfling(1980) or Korolyuk and Borovskikh(1994)) we find the so-called projection of the kernel, namely

$$\psi(x) = E[\Psi(X_1, X_2, X_3)|X_1 = x],\ x \in R^1,$$

The kernel belongs to the degenerate type if this projection equals to zero almost surely. Otherwise it is non-degenerate and then the theory is simpler, in particular U-statistics are asymptotically normal.

As the statistic M_n is clearly scale-free we may assume that the observations $X_1, \dots, X_n$ under H_0 are standard normal. We have

$$\begin{aligned}\psi(x) &= E_{X_2,X_3}\Psi(x, X_2, X_3)\\ &= \frac{1}{2} - \frac{1}{3}(2P(X_3\sqrt{2} - X_2 > x) + P(X_2 + X_3 < x\sqrt{2}))\\ &= \frac{1}{2} - \frac{2}{3}[1 - \Phi(x/\sqrt{3})] - \frac{1}{3}\Phi(x)\\ &= -\frac{1}{6} - \frac{1}{3}\Phi(x) + \frac{2}{3}\Phi(x/\sqrt{3}),\end{aligned}$$

where Φ is the standard normal d.f. Hence our kernel is non-degenerate.

To calculate the two first moments of $\psi(X)$ we need the values of some definite integrals.

LEMMA 1. For any real α

$$\begin{aligned}&\text{i)} \quad h_1(\alpha) = \int_{-\infty}^{+\infty} \Phi(\alpha x)d\Phi(x) = \frac{1}{2}\\ &\text{ii)} \quad h_2(\alpha) = \int_{-\infty}^{+\infty} \Phi^2(\alpha x)d\Phi(x) = \frac{1}{2} - \frac{1}{\pi}\arctan\frac{1}{\sqrt{1+2\alpha^2}}\\ &\text{iii)} \quad h_3(\alpha) = \int_{-\infty}^{+\infty} \Phi(\alpha x)\Phi(x)d\Phi(x) = \frac{1}{4} + \frac{1}{2\pi}\arctan\frac{\alpha}{\sqrt{2+\alpha^2}}.\end{aligned}$$

Proof: All three statements are proved using the differentiation with respect to α. Clearly

$$h_1'(\alpha) = \int_{-\infty}^{+\infty} x\varphi(\alpha x)d\Phi(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} x\exp(-(\alpha^2+1)x^2/2)dx = 0$$

because the integrand is an integrable odd function. Hence $h_1(\alpha) =$ const and does not depend on α. Substituting in $h_1(\alpha)$ the value $\alpha = 0$, we get i). Further, differentiating with respect to α and integrating by parts, we have

$$\begin{aligned}
h_2'(\alpha) &= \frac{1}{\pi}\int_{-\infty}^{+\infty} x\exp(-(\alpha^2+1)x^2/2)\Phi(\alpha x)dx \\
&= -\frac{1}{\pi(1+\alpha^2)}\int_{-\infty}^{+\infty} \Phi(\alpha x)d[\exp(-(\alpha^2+1)x^2/2)] \\
&= \frac{\alpha}{\pi(1+\alpha^2)}\int_{-\infty}^{+\infty} \exp(-(\alpha^2+1)x^2/2))\varphi(\alpha x)dx \\
&= \frac{\alpha}{\pi(1+\alpha^2)\sqrt{1+2\alpha^2}}\,.
\end{aligned}$$

Integrating this expression we obtain substituting $x = (\tan t)/\sqrt{2}$ for some constant C

$$\begin{aligned}
h_2(\alpha) + C &= \pi^{-1}\int \alpha d\alpha/(1+\alpha^2)\sqrt{1+2\alpha^2} \\
&= \pi^{-1}\int \tan t\cos t dt/(\cos^2 t(2+\tan^2 t)) \\
&= \pi^{-1}\int \sin t dt/(1+\cos^2 t) \\
&= -\pi^{-1}\arctan\frac{1}{\sqrt{1+2\alpha^2}}\,.
\end{aligned}$$

For $\alpha = 0$ we have $h_2(0) = \frac{1}{4}$, hence $C = -\frac{1}{2}$ and we obtain ii). Quite analogously

$$\begin{aligned}
h_3'(\alpha) &= \int_{-\infty}^{+\infty} x\varphi(x\alpha)\varphi(x)\Phi(x)dx \\
&= -(2\pi(1+\alpha^2))^{-1}\int_{-\infty}^{+\infty} \Phi(x)d[\exp(-x^2(1+\alpha^2)/2)] \\
&= (2\pi\sqrt{2\pi}(1+\alpha^2))^{-1}\int_{-\infty}^{+\infty} \exp(-x^2(2+\alpha^2)/2)dx \\
&= (2\pi(1+\alpha^2))^{-1}(2+\alpha^2)^{-1/2}\,.
\end{aligned}$$

Integrating in the same manner we have for some constant C_1

$$\begin{aligned}
C_1 + h_3(\alpha) &= (2\pi)^{-1}\int d\alpha/((1+\alpha^2)\sqrt{2+\alpha^2}) \\
&= (2\pi)^{-1}\int \cos t dt/(1+\sin^2 t) \\
&= (2\pi)^{-1}\arctan\frac{\alpha}{\sqrt{2+\alpha^2}}\,.
\end{aligned}$$

For $\alpha = 0$ we have $h_3(0) = \frac{1}{4}$, hence $C_1 = -\frac{1}{4}$ and we obtain iii). ■

Now we return to the calculation of the moments of $\psi(X)$. It follows from the statement i) of Lemma 1 that

$$E\psi(X) = -\frac{1}{6} - \frac{1}{3}\int_{-\infty}^{+\infty} \Phi(x)d\Phi(x) + \frac{2}{3}\int_{-\infty}^{+\infty} \Phi(x/\sqrt{3})d\Phi(x) = 0\,,$$

as it should be because $E\Psi(X, Y, Z) = 0$.

Now we calculate the variance $\Delta^2 = E\psi^2(X)$. We have after some simplifications

$$\Delta^2 = \frac{13}{36} - \frac{2}{9}h_1(1/\sqrt{3}) + \frac{4}{9}h_2(1/\sqrt{3}) - \frac{4}{9}h_3(1/\sqrt{3})\,.$$

Using the results i)-iii) of Lemma 1 we get finally

$$\Delta^2 = \frac{13}{108} - \frac{4}{9\pi}\left(\arctan\sqrt{\frac{3}{5}} + \frac{1}{2}\arctan\frac{1}{\sqrt{7}}\right) \approx 1.571236 \cdot 10^{-3} > 0\,.$$

Now we can apply the central limit theorem for non-degenerate U-statistics (see, e.g. Serfling (1980) or Korolyuk and Borovskikh (1994)). Consequently we have the convergence in distribution

$$\sqrt{n}M_n \to N(0, 9\Delta^2)\,.$$

This asymptotic result enables us to construct approximate one-sided and two-sided critical domains. Now we will touch on the consistency properties of the test based on statistic M_n.

Suppose that the alternative d.f. of X_i is $G(x)$. As the test statistic M_n obeys the law of large numbers for U-statistics (see, e.g. Korolyuk and Borovskikh(1994)), then under the alternative d.f. we have convergence in probability (and even almost surely)

$$M_n \to b(G) = E_G\Psi(X, Y, Z) = 1/2 - P_G(X + Y < Z\sqrt{2})\,,$$

where the random variables X, Y, Z have the d.f.G. Hence

$$b(G) = 1/2 - \int_{-\infty}^{+\infty} G * G(t\sqrt{2})\,dG(t)$$

where $*$ means the convolution operation. According to the general theory (see, e.g., Hettmansperger (1984), Section 1.3) our test is consistent for some class of consistency $\mathcal{G}$ of alternative d.f.'s G if $b(G) = 0$ under the null-hypothesis of normality with zero mean and $b(G) \neq 0$ for any $G \in \mathcal{G}$. It is possible to show that our test is consistent against the shifted and skew normal distributions as well as for the contamination alternative as defined, e.g., in Tukey (1960) and Huber (1981). We omit the proofs and prefer to explore the efficiency of the proposed test.

4. Efficiency

We can use for the description of the efficiency properties of our statistic both Pitman and Bahadur approaches which are rather close. On one hand, see Bahadur(1960), for asymptotically normal statistics Pitman efficiency usually coincides with the local approximate and exact Bahadur efficiency. On the other hand according to Wieand (1976) even for statistics with non-normal limiting distribution the local approximate Bahadur efficiency is equal under mild conditions to the so-called limiting Pitman efficiency. See Nikitin (1995) for the detailed definitions of efficiencies and their relationship.

We calculate in this paper the local exact Bahadur efficiency. However it can be proved along the lines of Rao(1965), Ch.7 that the values of Pitman efficiency are the same.

First we consider the simple alternative H_1' under which the d.f. of observations is $G(x) = \Phi((x - \theta)/\tau)$, where θ is some nonzero mean and $\tau^2 > 0$ is some known variance. If $\theta = 0$ we get the null hypothesis.

The measure of Bahadur efficiency is the Bahadur exact slope describing the rate of exponential decrease to zero of the attained level when the number of observations tends to infinity. It is well-known (see, e.g., Bahadur (1971) or Nikitin (1995)) that the calculation of the exact slope $c_T(\theta)$ of any sequence of statistics T_n depends on the large deviation behavior of this sequence under the null-hypothesis and the almost sure limit under the alternative. The latter is just the function $b(G)$ given above. To calculate it in our case we need a simple lemma.

LEMMA 2. For any real a, b and any positive l, m one has equality

$$\int_{-\infty}^{+\infty} \Phi((b-z)/m) d\Phi((z-a)/l) = \Phi((b-a)/\sqrt{l^2+m^2})$$

•

Proof: Let X and Y are independent standard normal variables. Then the right-hand and the left-hand sides are the expressions for the value of the d.f. for the r.v. $lX + a + mY$ in the point b. ■

In our case we get using Lemma 2

$$G * G(t\sqrt{2}) = \Phi((t - \sqrt{2}\theta)/\tau),$$

and consequently as $\theta \to 0$

$$\begin{aligned} b(G) &= \frac{1}{2} - \int_{R^1} G * G(t\sqrt{2}) dG(t) \\ &= \Phi((2-\sqrt{2})\theta)/2\tau) - 1/2 \sim (\sqrt{2}-1)\theta/2\tau\sqrt{\pi}. \end{aligned}$$

We can extract the large deviation asymptotics from recent papers by Nikitin and Ponikarov (1999a),(1999b). As our statistic M_n is non-degenerate U-statistic we get from these papers that

$$\lim_{n\to\infty} n^{-1} \ln P(M_n \geq t) = -h(t),$$

where the function h is continuous for sufficiently small t and, moreover, as $t \to 0$ one has

$$h(t) = t^2/(18\Delta^2)(1 + o(1)).$$

Combining this result with the asymptotics we got above for $b(G)$ we get easily from Bahadur (1971) that the exact slope of the sequence M_n admits the representation as $\theta \to 0$:

$$c_M(\theta) = 2h(b(G)) \sim (\sqrt{2}-1)^2\theta^2/(36\tau^2\pi\Delta^2).$$

To understand if this value is high enough we must calculate the upper bound for the exact slope given by the so-called Bahadur-Raghavachari inequality (see Bahadur (1971), Section 7 and Nikitin

(1995), Ch.1). According to that inequality and taking in consideration that the hypothesis H_0 is composite (the value of σ is not specified) we get that

$$c_M(\theta) \leq 2K(\theta, \tau),$$

where

$$K(\theta, \tau) = \inf_{\sigma>0} \int_{R^1} \ln[\varphi(x, \theta, \tau)/\varphi(x, 0, \sigma)]\varphi(x, \theta, \tau)dx,$$

where $\varphi(x, \theta, \sigma)$ is the normal density with mean θ and standard deviation σ. The integral is calculated easily and its value is

$$\ln(\sigma/\tau) + (\tau^2 + \theta^2)/2\sigma^2 - 1/2.$$

The infimum of this expression is attained at $\sigma_0^2 = \tau^2 + \theta^2$ and is equal to

$$K(\theta, \tau) = \frac{1}{2}\ln(1 + \theta^2/\tau^2) \sim \theta^2/2\tau^2, \text{ as } \theta \to 0.$$

It is natural to define the local Bahadur efficiency as

$$e_M^B = \lim_{\theta\to 0} c_M(\theta)/2K(\theta, \tau).$$

Using the calculations made above we get

$$e_M^B = (\sqrt{2} - 1)^2/(36\pi\Delta^2) \approx 0.9655\ldots$$

This value of efficiency is very high and this increases the interest to the proposed test.

Another simple alternative to the normal distribution with zero mean is the skew normal distribution introduced by Azzalini (1985) and later studied by many authors, see, e.g. , Henze(1986), Azzalini and Dalla Valle (1996), Chiogna (1998) and Pewsey (2000).

In this case the density g of the alternative d.f. G is

$$g(x) = 2\varphi(x)\Phi(\theta x), x \in R^1, \theta \in R^1.$$

For simplicity we take the scale factor equal to one. The following calculations can be made with arbitrary scale factor τ and the result will be the same. The main difference with the previous case consists in the form of $b(G)$ and of the upper bound. Using the Taylor expansion valid for any x

$$2\Phi(\theta x) = 1 + 2x\varphi(0)\theta + \theta^2 x^2 \varphi'(\xi x),\ 0 < \xi < \theta\,,$$

we get

$$g * g(z) = \int_{R^1} \varphi(z-y)\varphi(y)[2\Phi(\theta(z-y))][2\Phi(\theta y)]dy\,.$$

Now we substitute in this integral the expansion for $2\Phi(\theta x)$, integrate it over $(-\infty, x\sqrt{2})$ in order to get $G * G(x\sqrt{2})$ and retain the terms only up to the first order in θ. It is clear that the remainder will be of order $O(\theta^2)$ uniformly in x due to the presence of strongly decreasing function φ in the integrand. We get after some calculations that as $\theta \to 0$

$$G * G(x\sqrt{2}) = \Phi(x) - 2\theta\varphi(x)/\sqrt{\pi} + O(\theta^2)$$

and hence

$$b(G) = 1/2 - \int_{-\infty}^{+\infty} G * G(x\sqrt{2})dG(x) = [(2-\sqrt{2})/2\pi]\theta + O(\theta^2)\,.$$

Consequently in this case we have

$$c_M(\theta) \sim (\sqrt{2}-1)^2\theta^2/(18\pi^2\Delta^2)\,.$$

The upper bound (we take again the scale factor as 1) is equal to

$$K(\theta, 1) = 2\inf_{\sigma>0}\int_{R^1} \ln[2\varphi(x,0,1)\Phi(x\theta)/\varphi(x,0,\sigma)]\varphi(x,0,1)\Phi(x\theta)dx\,.$$

This integral is easy analysed if one uses the expansion of $\Phi(x\theta)$ as above. The infimum is attained for $\sigma^2 = 1 + O(\theta^2)$ and hence

$$2K(\theta) \sim 4\varphi^2(0)EX_1^2\theta^2 \sim 2\theta^2/\pi\,.$$

Comparing the upper bound with the local exact slope given below we see that the efficiency is the same as in the location case, namely $e_M^B \approx 0.9655\ldots$

This coincidence is not surprising because there was found in Durio and Nikitin (2001) that such equality takes place also for many classical nonparametric statistics of Kolmogorov-Smirnov and ω^2 type and that this is the characteristic property of the normal law for such statistics.

5. Conclusion

In this paper we introduced a new scale-free statistic for testing normality with zero mean which has a simple structure and is asymptotically normal under the null-hypothesis. We described sufficient conditions for the consistency of corresponding test and found local Bahadur efficiency for normal shift and skew alternatives which turned out to be high.

It would be interesting to explore the properties of our test for different alternatives. Another problem of interest is the asymptotic comparison of the new test with other tests of normality, say, with the Shapiro – Wilk test (see Shapiro and Wilk (1965)). It is known from Stephens (1974) that this test has a good power but nothing is presently known on his efficiency as well as on the efficiency of other omnibus tests of normality.

REFERENCES

Ahmad, I. and Alwasel, I. (1999) A goodness-of-fit test for exponentiality based on the memoryless property, *J.R. Statist. Soc.*, B61, Pt.3, 681-689.

Angus, J.E. (1982) Goodness-of-fit tests for exponentiality based on a loss-of-memory type functional equation, *Journ. Statist. Plann. Inference*, 6, 241-251.

Azzalini, A. (1985) A class of distributions which includes the normal ones, *Scand. J. Stat.*, 12, 171-178.

Azzalini, A. and Dalla Valle, A. (1996) The multivariate skew-normal distribution, *Biometrika*, 83, 715-726.

Bahadur, R.R. (1960) Stochastic comparison of tests, *Ann. Math. Stat.*, 31, 276-295.

Baringhaus, L. and Henze, N. (1992) A characterization of and new consistent tests of symmetry, *Commun. Statist. Theor. Meth.*, 21, 1555-1566.

Bryc, W. (1995) Normal Distribution: Characterization with Applications, *Lecture Notes in Statistics*, vol. 100, Springer, Berlin.

Chiogna, M. (1998) Some results on the scalar skew-normal distribution, *J. Ital. Statist. Soc.*, 7, 1-13.

Durio, A. and Nikitin, Ya.Yu. (2001) *Local asymptotic efficiency of some goodness-of-fit tests under skew alternative*, Submitted to publication.

Eichelsbacher, P. and Loewe, M. (1995) A large deviation principle for m-variate von Mises-statistics and U-statistics, *J. of Theor. Prob.*, 8, 807-824.

Henze, N. (1986) A probabilistic representation of the "skew-normal" distribution, *Scand. J. Stat.*, 13, 271-275.

HETTMANSPERGER, T. (1984) *Statistical Inference Based on Ranks*, Wiley, New York.

HOLLANDER, M. and PROSCHAN, F. (1972) Testing whether new is better than used, *Ann. Math. Stat.*, 43, 1136-1146.

HUBER, P.J. (1981) *Robust Statistics*, Wiley, New York.

JANSSEN, P.L. (1988) *Generalized Empirical Distribution Functions with Statistical Applications*, Limburgs Universitair Centrum, Diepenbeek.

KAGAN, A.M., LINNIK, YU.V., and RAO, C.R. (1973) *Characterization Theorems of Mathematical Statistics*, Wiley, New York.

KOROLYUK, V.S. and BOROVSKIKH, YU.V. (1994) *Theory of U-statistics*, Kluwer, Dordrecht.

KOUL, H.L. (1977) A test for new better than used, *Commun.Statist. Theory Meth.*, A6, 563-574.

MATHAI, A.M. and PEDERZOLI, G. (1977) *Characterizations of the Normal Probability Law*, Wiley, New York.

NIKITIN, YA. (1995) *Asymptotic Efficiency of Nonparametric Tests*, Cambridge University Press.

NIKITIN, YA. and PONIKAROV E.V. (1999a) Large deviations of Chernoff type for U-statistics and von Mises functionals, *Proc. of St. Petersburg Math. Soc.*, 7, 124-167, in Russian. To be translated by Amer. Math. Soc.

NIKITIN, YA. and PONIKAROV E.V. (1999b) Large deviations of Chernoff type for U- and V-statistics, *Doklady RAN*, 369, 13-16.

PEWSEY, A. (2000) Problems of inference for Azzalini's skew-normal distribution, *J. of Appl. Stat.*, 27, 859-870.

POLYA, G.(1923) Herleitung des Gauss'schen Fehlergesetzes aus einer Funktionsalsgleichung, *Math. Zeitschrift*, 18, 96-108.

RAO, C.R. (1965) *Linear Statistical Inference and its Applications*, Wiley, New York.

SERFLING, R.J. (1980) *Approximation Theorems of Mathematical Statistics*, Wiley, New York.

SHAPIRO, S.S. and WILK M.B. (1965) An analysis of variance test for normality (complete samples), *Biometrika*, 52, 591-611.

STEPHENS, M.A. (1974) EDF statistics for goodness of fit and some comparisons, *Journ. Amer. Stat. Ass.*, 69, 730-737.

TUKEY, J.W.(1960) A survey of sampling from contaminated distributions, *In: Contributions to Probability and Statistics, Olkin,I. et al., eds*, (Stanford University Press), 448-485.

WIEAND, H.S. (1976) A condition under which the Pitman and Bahadur approaches to efficiency coincide, *Ann. Statist.*, 4, 1003-1011.

Scale-invariant test of normality based on Polya's characterization

Summary

We propose a new scale-invariant test of normality based on Polya's characterization.The test statistic is a non-degenerate U-statistic. We find the local Bahadur efficiency of our test against the normal location and skew alternatives. This efficiency turns out to be rather high.

Un test di normalità, invariante per trasformazioni di scala, basato sulla caratterizzazione di Polya

Riassunto

In questo lavoro proponiamo un test di normalità, invariante per trasformazioni di scala, basato sulla caratterizzazione di Polya. Il test proposto è una U-Statistic non degenere. Nel lavoro ne studiamo le proprietà e determiniamo l'efficienza locale del test secondo l'impostazione di Bahadur. Le ipotesi alternative considerate sono quelle di posizione e di distribuzione asimmetrica. Per entrambe le alternative l'efficienza determinata è alta.

Key words

Characterization of normality; U-statistics; Bahadur efficiency; Skew normal distribution.

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