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On Quasi-Means

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Abstract. We generalize the notion of mean value by considering functions of the form:

$$M(x_1, \ldots, x_n) = f^{-1}\left[\sum_{i=1}^{n} p_i g(x_i)\right].$$

We term these functions quasi-means, and we propose a characterization based on two associative operations—corresponding respectively to combination of individual coordinates and combinations of aggregate values. We then show that the only homogeneous quasi-means are quasi-linear, and discuss a further extension.

1. Introduction

Let $M : R^n \to R$, and let $*$ and $\circ$ be binary operations. The functional equation:

$$M(x_1 * y_1, \ldots, x_n * y_n) = M(x_1, \ldots, x_n) \circ M(y_1, \ldots, y_n) \tag{1}$$

arises when it is desired to compute summary or aggregate quantities, and the operation which is deemed appropriate for combining individual coordinates does not necessarily coincides with that deemed appropriate for combining aggregate values. In this paper we investigate the nature of functions M satisfying (1).

2. A Characterization Result

A binary operation $\circ$ is associative whenever $(x \circ y) \circ z = x \circ (y \circ z)$, for all $x, y, z \in I$ where I is a proper interval of real numbers (closed, open, half open, finite or infinite). Also, $\circ$ is cancellative if $x \circ y = x \circ z$ or $y \circ x = z \circ x$ imply $z = y$, for all $x, y, z \in I$. Aczél (1966, pp. 253 ff., or 1987, pp. 106 ff.) characterizes the continuous, associative and cancellative operations as those satisfying:

$$x \circ y = f^{-1}[f(x) + f(y)], \tag{2}$$

for all $x, y \in I$, where $f : I \to J$ is continuous, strictly monotone, and unique up to a positive affine transformation, and J is one of the real intervals $(-\infty, \alpha]$,

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$(-\infty, \alpha)$, $[\beta, \infty)$, (β, ∞), $(-\infty, \infty)$, $\alpha \leq 0$, $\beta \geq 0$. The operation $\circ$ is not assumed to be symmetric but it is so as a result of (2).

Let f and g be two functions with domain I and range J, of the form described in the previous paragraph and let:

$$x * y = g^{-1}[g(x) + g(y)] \tag{3}$$

$$x \circ y = f^{-1}[f(x) + f(y)]. \tag{4}$$

Theorem 1. *The function $M : R^n \to R$ satifies the equation:*

$$M(x_1 * y_1, \dots, x_n * y_n) = M(x_1, \dots, x_n) \circ M(y_1, \dots, y_n) \tag{5}$$

and is continuous at a point or bounded on one side or an interval or on a set of positive measure, if and only if

$$M(x_1, \dots, x_n) = f^{-1}\left[\sum_{i=1}^{n} p_i g(x_i)\right] \tag{6}$$

where f and g are two continuous and strictly monotone functions, generated by a binary associative operation, and where $p_1, \dots, p_n$ are real constants.

Proof: Substituting (3) and (4) into (5), and defining

$$f(M(x_1, \dots, x_n)) = Z(x_1, \dots, x_n),$$

we have:

$$Z(g^{-1}(g(x_1)+g(y_1)), \dots, g^{-1}(g(x_n)+g(y_n))) = Z(x_1, \dots, x_n) + Z(y_1, \dots, y_n). \tag{7}$$

Setting $g(x_i) = u_i$ and $g(y_i) = v_i$ in (7), and defining $Z(g^{-1}(\cdot), \dots, g^{-1}(\cdot)) = H(\cdot, \dots, \cdot)$, we obtain:

$$H(u_1 + v_1, \dots, u_n + v_n) = H(u_1, \dots, u_n) + H(v_1, \dots, v_n). \tag{8}$$

By Aczél (1966, pp. 215–216 and p. 32) the general solution of (8), continuous at a point or bounded on one side or an interval or on a set of positive measure, is:

$$H(u_1, \dots, u_n) = p_1 u_1 + \dots + p_n u_n, \tag{9}$$

with $p_1, \dots, p_n$ real constants. Hence the general solution of (5), under the same regularity conditions, is (6). ∎

In the remainder of the paper we term functions satisfying (5) quasi-means.

Example 1: Take $x \circ y = (x^m + y^m)^{1/m}$ and $x * y = x + y + xy$. Then $f(x) = x^m$ and $g(x) = \log(x+1)$, $x > -1$. Therefore the quasi-mean must be:

$$\left[\sum_{i=1}^{n} p_i \log(x_i + 1)\right]^{\frac{1}{m}} .$$

When $m = 1$, the resulting quasi-mean has found application in the context of the measurement of welfare (see Tinbergen, 1991).

Example 2: Take $x \circ y = x + y$ and $x * y = xy$. Then $f(x) = x$ and $g(x) = \log x$. The quasi-mean

$$M(x_1, \ldots, x_n) = \sum_{i=1}^{n} p_i \log x_i$$

is the celebrated utility function proposed by Daniel Bernoulli (1738). If instead $x * y = (x^m + y^m)^{1/m}$, $g(x) = x^m$ and the quasi-mean is

$$M(x_1, \ldots, x_n) = \sum_{i=1}^{n} p_i x_i^m$$

with applications in various fields such as mathematical statistics (moments), economics (production functions) and social choice theory (aggregation functions).

Example 3: Take $x \circ y = x + y$ and $\sum_{i=1}^{n} p_i = 1$. Then the quasi-mean is the expected value of the function g of a random variable taking values x_i with probability p_i.

The mapping $M : R^n \to R$ is a quasilinear mean if there exists a continuous strictly monotone function $f : R \to R$ and real constants $p_1, \ldots, p_n$ with $\sum_{i=1}^{n} p_i = 1$, such that:

$$M(x_1, \ldots, x_n) = f^{-1}\left[\sum_{i=1}^{n} p_i f(x_i)\right] . \qquad (10)$$

The following Corollaries are given without proof.

Corollary 1. *A characterization of the quasilinear mean (10) is given by:*

$$M(x_1 \circ y_1, \ldots, x_n \circ y_n) = M(x_1, \ldots, x_n) \circ M(y_1, \ldots, y_n) \qquad (11)$$

and $M(x_1, \ldots, x_n)$ is reflexive.

Characterizations of quasilinear mean have been given by de Finetti (1931), Kitagawa (1934) and Aczél (1948).

Corollary 2. *A characterization of the symmetric quasilinear mean:*

$$M(x_1, \ldots, x_n) = f^{-1}\left[\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right]$$

is given by (11) and $M(x_1, \ldots, x_n)$ *is reflexive and symmetric.*

Kolmogorov (1930), Nagumo (1930), de Finetti (1931) and Aczél (1948) have given necessary and sufficient conditions for a function M to be a symmetric quasilinear mean.

Example 4: Let M be symmetric and reflexive, and $x*y = x \circ y$. Typical examples of means that can be characterized by the functional equation (11) are:

Arithmetic mean	$x \circ y = x + y$	$f(x) = x$
Geometric mean	$x \circ y = xy$	$f(x) = \log x,\ x > 0$
Harmonic mean	$x \circ y = xy/(x+y)$	$f(x) = 1/x,\ x > 0$
Root-mean-power	$x \circ y = (x^m + y^m)^{1/m}$	$f(x) = x^m,\ x > 0,\ m \neq 0$

This result was previously obtained by Matsumara (1933) for the arithmetic mean (see also Aczél, 1966, p. 239), and by Nakahara (1936) for the geometric mean.

3. Invariance Properties

Homogeneous quasi-means satisfy:

$$M\{\lambda x_1, \ldots, \lambda x_n\} = \lambda M\{x_1, \ldots, x_n\} \tag{12}$$

The following result provides a characterization of homogeneous quasi-means.

Theorem 2. *Let M be a quasi-mean, as defined by (6). If M is homogeneous, then*

i) $f(x) = g(x)$;

ii) $g(x) = \alpha x^m + \beta, x > 0, m \neq 0$ *or* $g(x) = \alpha \log x + \beta, \alpha \neq 0$.

Conversely, if $f(x) = g(x) = \alpha x^m + \beta,\ x > 0,\ m \neq 0$, then M is homogeneous.

Proof: From homogeneity,

$$\lambda f^{-1}\left[\sum_{i=1}^{n} p_i g(x_i)\right] = f^{-1}\left[\sum_{i=1}^{n} p_i g(\lambda x_i)\right] \tag{13}$$

with $\lambda > 0$. Define:

$$f(\lambda x) = f_\lambda(x)$$
$$g(x_i) = u_i$$

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with $\lambda > 0$. Define:

$$f(\lambda x) = f_\lambda(x)$$
$$g(x_i) = u_i$$

Since f is strictly monotone, this gives:

$$f_\lambda \left[f^{-1}\left(\sum_{i=1}^{n} p_i u_i \right) \right] = \sum_{i=1}^{n} p_i g_\lambda (g^{-1}(u_i)). \qquad (14)$$

Moreover, using the notation:

$$H_\lambda(y) = f_\lambda\left(f^{-1}(y)\right)$$
$$Z_\lambda(u_i) = g_\lambda\left(g^{-1}(u_i)\right),$$

(13) becomes:

$$H_\lambda \left[\sum_{i=1}^{n} p_i u_i \right] = \sum_{i=1}^{n} p_i Z_\lambda(u_i) \qquad (15)$$

with $u_i \in g(I) \equiv J$. Define $W = \times_{i=1}^{n} J \in R^n$, and take an arbitrary inner point $(w_0, \ldots, w_0)$. Write $(u_1, \ldots, u_n) = (w_0 + w_1, \ldots, w_0 + w_n)$ with $w_0 + w_i \in J'$. Then, (15) becomes:

$$H_\lambda \left[w_0 \sum_{i=1}^{n} p_i + \sum_{i=1}^{n} p_i w_i \right] = \sum_{i=1}^{n} p_i Z_\lambda(w_0 + w_i). \qquad (16)$$

Put now successively $w_k = 0,\ k = 1, \ldots, i-1, i+1, \ldots, n$, to obtain:

$$p_i Z_\lambda(w_0 + w_i) = H_\lambda \left[w_0 \sum_{i=1}^{n} p_i + p_i w_i \right] - c \quad i = 1, \ldots, n \qquad (17)$$

where c does not depend on the w_i. Inserting (17) into (16) and defining:

$$D_\lambda(y) = H_\lambda \left[w_0 \sum_{i=1}^{n} p_i + y \right] - c$$
$$D_\lambda^i(y) = \frac{1}{p_i} D_\lambda(p_i y) \quad i = 1, \ldots, n$$

we obtain:

$$D_\lambda(w_1 + \cdots + w_n) = D_\lambda^1(w_1) + \cdots + D_\lambda^n(w_n), \qquad (18)$$

Following Eichhorn (1978, pp. 33–34), it is possible to determine the solution of (18), and deduce that g must satisfy an equation like:

$$g(\lambda x) = a(\lambda) g(x) + b(\lambda).$$

It is known (see Hardy, Littlewood and Pólya, 1934, pp. 68) that there exist two system of solutions to this equation, namely:

$$g(x) = \alpha x^m + \beta,\ x > 0,\ m \neq 0,\ \alpha \neq 0$$
$$g(x) = \alpha \log x + \beta,\ x > 0,\ \alpha \neq 0.$$

To solve for f, let now:

$$M_g(x_1, \ldots, x_n) = g^{-1}\left[\sum_{i=1}^{n} p_i g(x_i)\right].$$

Defining $f^{-1}[g(x)] = \phi(x)$, (13) becomes:

$$\lambda\phi[M_g(x_1, \ldots, x_n)] = \phi[M_g(\lambda x_1, \ldots, \lambda x_n)].$$

Since M_g is homogeneous, we have:

$$\lambda\phi[M_g(x_1, \ldots, x_n)] = \phi[\lambda M_g(x_1, \ldots, x_n)]$$

for all $(x_1, \ldots, x_n)$. Therefore $\phi(x) = x$ and $f = g$. This concludes the proof. ∎

Remark: If M is a mean defined by (6), the condition $f(x) = g(x) = \alpha \log x + \beta$, $\alpha \neq 0$ is sufficient to prove homogeneity of M only if $\sum_{i=1}^{n} p_i = 1$. Examples where $\sum_{i=1}^{n} p_i \neq 1$ and M is not homogeneous are easily constructed.

Example 5: In the theory of production functions, adopting a quasi-mean instead of a quasi-linear function relaxes the restriction that inputs and outputs must be combined according to the same associative operation. However, if homogeneity is desired, the two operations must be equal, and the form has to be as in Theorem 2. In particular, $f(x) = g(x) = x^{-m}$ implies an ACMS production function:

$$M\{x_1, \ldots, x_n\} = \left(p_1 x_1^{-m} + \cdots + p_n x_n^{-m}\right)^{-\frac{1}{m}}$$

with $p_1 > 0, \ldots, p_n > 0$ (see Eichhorn, 1978, p. 31). On the other hand, $f(x) = g(x) = \log x$ and $\sum_{i=1}^{n} p_i = 1$ yield a Cobb-Douglas production function:

$$M\{x_1, \ldots, x_n\} = \exp\{p_1 \log x_1 + \cdots + p_n \log x_n\}.$$

The next invariance condition we study is traslativity. Translative quasi-means obey the condition:

$$M\{x_1 + t, \ldots, x_n + t\} = M\{x_1, \ldots, x_n\} + t \tag{19}$$

Similarly to homogeneity, we have the following result.

Theorem 3. *A mean satisfying (5) is translative if and only if* $f(x+c) = g(x)$ *and:* $g(x) = \alpha x + \beta, x > 0$ *or* $g(x) = \alpha \exp\{\gamma x\} + \beta$, *with* $\alpha \neq 0, \gamma \neq 0$.

Proof: Following the same steps as in the Proof of Theorem 2, it is possible to show that g must satisfy:

$$g(x+t) = a(t)g(x) + b(t) \tag{20}$$

from which the required form follows. Define

$$M_g(x_1, \ldots, x_n) = g^{-1}\left[\sum_{i=1}^{n} p_i g(x_i)\right].$$

Translativity requires:

$$\phi[M_g(x_1, \ldots, x_n)] + t = \phi[M_g(x_1 + t, \ldots, x_n + t)],$$

where $f^{-1}[g(x)] = \phi(x)$. Since M_g is translative, we have:

$$\phi[M_g(x_1, \ldots, x_n)] + t = \phi[M_g(x_1, \ldots, x_n) + t]$$

for all $(x_1, \ldots, x_n)$. Therefore $\phi(x) = x + c$, and the theorem is proved. ∎

The arithmetic mean is then the only homogeneous and translative mean satisfying (5).

Example 6: A translative mean with $f \neq g$ can be obtained by setting $x * y = \log(\exp\{x\} + \exp\{y\})$, $x \circ y = \log(\exp\{x+c\} + \exp\{y+c\}) - c$, so that $g(x) = \exp\{x\}$ and $f(x) = \exp\{x+c\}$.

Finally, we remark that two quasi-means M_1 and M_2, with functions f_1, g_1 and f_2, g_2 are equal for all $(x_1, \ldots, x_n)$ whenever $f_1(x) = \alpha f_2(x) + \beta$ and $g_1(x) = \alpha g_2(x) + \beta$. A proof of this statement follows the lines of the proof of Theorem 2.

4. A Further Extension

A further characterization can be obtained by relaxing the restriction that, in the functional equation (5), the quasi-mean should be the same.

Theorem 4. *The functions* $M_i : R^n \to R, i = 1, 2, 3$ *satisfy the equation:*

$$M_1(x_1 * y_1, \ldots, x_n * y_n) = M_2(x_1, \ldots, x_n) \circ M_3(y_1, \ldots, y_n) \tag{21}$$

and are continuous at a point or bounded on one side or an interval or on a set of positive measure, if and only if:

$$M_1(x_1,\ldots,x_n) = f^{-1}\left[\sum_{i=1}^{n} p_i g(x_i) + a + b\right] \tag{22}$$

$$M_2(x_1,\ldots,x_n) = f^{-1}\left[\sum_{i=1}^{n} p_i g(x_i) + a\right] \tag{23}$$

$$M_3(x_1,\ldots,x_n) = f^{-1}\left[\sum_{i=1}^{n} p_i g(x_i) + b\right] \tag{24}$$

where $p_1,\ldots,p_n$, a *and* b *are real constants.*

Proof: Substituting (3) and (4) into (21), and defining

$$f(M_i(x_1,\ldots,x_n)) = Z_i(x_1,\ldots,x_n),$$

we have:

$$Z_1(g^{-1}(g(x_1)+g(y_1)),\ldots,g^{-1}(g(x_n)+g(y_n)))=Z_2(x_1,\ldots,x_n)+Z_3(y_1,\ldots,y_n) \tag{25}$$

Setting $g(x_i) = u_i$ and $g(y_i) = v_i$ in (25), and defining $Z_i(g^{-1}(\cdot),\ldots,g^{-1}(\cdot)) = H_i(\cdot,\ldots,\cdot)$, we obtain:

$$H_1(u_1+v_1,\ldots,u_n+v_n) = H_2(u_1,\ldots,u_n) + H_3(v_1,\ldots,v_n). \tag{26}$$

By Aczél (1966, pp. 348) the general solution of (26), continuous at a point or bounded on one side or an interval or on a set of positive measure, is:

$$H_1(u_1,\ldots,u_n) = p_1u_1 + \cdots + p_nu_n + a + b \tag{27}$$

$$H_2(u_1,\ldots,u_n) = p_1u_1 + \cdots + p_nu_n + a \tag{28}$$

$$H_3(u_1,\ldots,u_n) = p_1u_1 + \cdots + p_nu_n + b \tag{29}$$

with $p_1,\ldots,p_n,a,b$ real constants. Hence the general solution of (21), under the same regularity conditions, is (22), (23), (24). ∎

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